

THE UNIRATIONALITY OF THE MODULI SPACES OF 2-ELEMENTARY $K3$ SURFACES

SHOUHEI MA

with an Appendix by Ken-Ichi Yoshikawa

ABSTRACT. We prove that the moduli spaces of $K3$ surfaces with non-symplectic involutions are unirational. As a by-product we describe configuration spaces of $5 \leq d \leq 8$ points in \mathbb{P}^2 as arithmetic quotients of type IV.

1. INTRODUCTION

$K3$ surfaces with non-symplectic involutions were classified by Nikulin [31], and Yoshikawa [36] showed that their moduli spaces are Zariski open sets of certain modular varieties of orthogonal type. In this paper we prove that those moduli spaces are unirational. This work was inspired by a recent result of Yoshikawa on the Kodaira dimensions of those spaces, which is presented by him in the Appendix A of this paper. Let us begin by recalling basic definitions.

Let X be a complex $K3$ surface with an involution ι . When ι acts nontrivially on $H^0(K_X)$, ι is called *non-symplectic*, and the pair (X, ι) is called a *2-elementary $K3$ surface*. In this case, the lattice $L_+ = H^2(X, \mathbb{Z})^\iota$ of ι -invariant cycles is a hyperbolic lattice with 2-elementary discriminant form D_{L_+} . The *main invariant* of (X, ι) is the triplet (r, a, δ) where r is the rank of L_+ , a is the length of D_{L_+} , i.e., $D_{L_+} \simeq (\mathbb{Z}/2\mathbb{Z})^a$, and δ is the parity of D_{L_+} . Nikulin [31] proved that the deformation type of (X, ι) is determined by the main invariant (r, a, δ) , and he enumerated all main invariants of 2-elementary $K3$ surfaces, which are seventy-five in number.

By the theory of period mapping, 2-elementary $K3$ surfaces of a fixed main invariant (r, a, δ) are parametrized by the Hermitian symmetric domain associated to a certain lattice L_- of signature $(2, 20 - r)$. Yoshikawa [36], [38] determined the correct monodromy group as the orthogonal group $O(L_-)$ of L_- . Consequently, he constructed the moduli space $\mathcal{M}_{(r,a,\delta)}$ of those pairs (X, ι) as a Zariski open set of the modular variety defined by $O(L_-)$.

The principal result of the present paper is the following.

Theorem 1.1. *For every main invariant (r, a, δ) the moduli space $\mathcal{M}_{(r,a,\delta)}$ of 2-elementary $K3$ surfaces of type (r, a, δ) is unirational.*

We recall that the 2-elementary $K3$ surfaces in $\mathcal{M}_{(1,1,1)}$ are double planes ramified over smooth sextics so that $\mathcal{M}_{(1,1,1)}$ is birational to the orbit space

2000 *Mathematics Subject Classification.* Primary 14J28, Secondary 14G35, 14H50.

Key words and phrases. $K3$ surface, non-symplectic involution, unirationality of moduli space, orthogonal modular variety, point set in projective plane.

$|\mathcal{O}_{\mathbb{P}^2}(6)|/\mathrm{PGL}_3$, which is unirational. This fact is a prototype of Theorem 1.1. Kondō [21] proved the rationality of $\mathcal{M}_{(10,2,0)}$ and $\mathcal{M}_{(10,10,0)}$, the latter being isomorphic to the moduli of Enriques surfaces. Shepherd-Barron [34] practically established the rationality of $\mathcal{M}_{(5,5,1)}$ in the course of proving that of the moduli of genus 6 curves. Matsumoto-Sasaki-Yoshida [26] constructed general members of $\mathcal{M}_{(16,6,1)}$ starting from six lines on \mathbb{P}^2 . A similar idea was used by Koike-Shiga-Takayama-Tsutsui [20] to obtain general members of $\mathcal{M}_{(14,8,1)}$ from four bidegree $(1, 1)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, $\mathcal{M}_{(16,6,1)}$ and $\mathcal{M}_{(14,8,1)}$ are also unirational.

Yoshikawa studied the birational type of $\mathcal{M}_{(r,a,\delta)}$ in a systematic way by using a criterion of Gritsenko [12] and Borchers products. He found that $\mathcal{M}_{(r,a,\delta)}$ has Kodaira dimension $-\infty$ when $13 \leq r \leq 17$ and when $r + a = 22$, $r \leq 17$. After that he suggested to the author to study the birational type of $\mathcal{M}_{(r,a,\delta)}$ through a geometric approach. The present work grew out of this suggestion. After Theorem 1.1 was proved, Yoshikawa and the author decided to write both approaches in this paper. Yoshikawa's work is presented in the Appendix A. Now the Kodaira dimensions of some of $\mathcal{M}_{(r,a,\delta)}$ may be calculated by two methods: by modular forms on the moduli spaces, and by the geometry of 2-elementary $K3$ surfaces.

We will prove Theorem 1.1 by using certain Galois covers of $\mathcal{M}_{(r,a,\delta)}$ and isogenies between them. The strategy is as follows.

- (1) Let $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ be the modular variety associated to the group $\widetilde{\mathrm{O}}(L_-)$ of isometries of L_- which act trivially on the discriminant form. The variety $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ is a Galois cover of $\mathcal{M}_{(r,a,\delta)}$.
- (2) Construct an isogeny $\widetilde{\mathcal{M}}_{(r,a,\delta)} \rightarrow \widetilde{\mathcal{M}}_{(r,a',\delta')}$ when $a' < a$, $\delta = 1$, and when $a' < a$, $\delta = \delta'$.
- (3) For each fixed r , choose a large a and find a moduli interpretation of (an open set of) $\widetilde{\mathcal{M}}_{(r,a,\delta)}$. Then prove that $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ is unirational using that interpretation. By step (2) follows the unirationality of $\widetilde{\mathcal{M}}_{(r,a',\delta')}$ for $a' < a$.
- (4) The remaining moduli spaces $\mathcal{M}_{(r,a'',\delta'')}$ with $a'' > a$, if any, are also proved to be unirational in some way.

One of the advantages of studying the covers $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ is that we have isogenies between them so that the problem is reduced to fewer modular varieties. Those isogenies admit geometric interpretation in terms of twisted Fourier-Mukai partners. By this strategy we will derive the unirationality of seventy $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ by studying just twenty-two $\widetilde{\mathcal{M}}_{(r,a,\delta)}$. The remaining five moduli spaces $\mathcal{M}_{(r,a,\delta)}$, for which we do not know whether the covers $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ are unirational, are treated in step (4) or already settled ([21]). In step (3), we often identify $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ with the moduli of certain plane sextics endowed with a labeling of the singularities. We can attach such geometric interpretations to $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ in a fairly uniform manner: this is another virtue of studying $\widetilde{\mathcal{M}}_{(r,a,\delta)}$. We shall explain a general idea of such interpretations (Section 3.4), discuss few cases in detail as models (Sections 4 and 5), and for other cases omit some detail.

Let us comment on other possible approaches for Theorem 1.1. Firstly, as explained by Alexeev-Nikulin [1], 2-elementary $K3$ surfaces with $r + a \leq 20$ are

related to log del Pezzo surfaces of index ≤ 2 . Thus one might study $\mathcal{M}_{(r,a,\delta)}$ via the moduli of such surfaces, using the explicit description of log del Pezzo surfaces of index 2 given by Nakayama [29]. Secondly, by using singular curves on \mathbb{P}^2 and \mathbb{F}_n as branches (as in this paper), for most (r, a, δ) we can actually find a unirational parameter space that dominates $\mathcal{M}_{(r,a,\delta)}$.

In [25], those will be developed further to derive the rationality of sixty-seven $\mathcal{M}_{(r,a,\delta)}$. Hence one may establish Theorem 1.1 also by just studying the remaining moduli spaces. However, the proof of rationality is delicate and ad hoc, so that the whole proof of unirationality would be lengthy if we do so. We here prefer the proof using $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ because it is more systematic, short, and self-contained.

We will relate the covers $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ with $r + a = 22$ and $r \geq 12$ to configuration spaces of points in \mathbb{P}^2 . As a by-product we describe those spaces as arithmetic quotients of type IV. To be more precise, let $U_d \subset (\mathbb{P}^2)^d$ (resp. $V_d \subset (\mathbb{P}^2)^d$) be the variety of d ordered points of which no three are collinear (resp. only the first three are collinear). Let U_d/G and V_d/G denote the quotient varieties for the diagonal actions of $G = \mathrm{PGL}_3$. Let L_n be the lattice $\langle 2 \rangle^2 \oplus \langle -2 \rangle^n$.

Theorem 1.2. *Let $5 \leq d \leq 8$. For each $1 \leq n \leq 8$ there exists an arithmetic group $\Gamma_n \subset \mathrm{O}(L_n)$ such that one has birational period maps*

$$U_d/G \dashrightarrow \mathcal{F}(\Gamma_{2d-8}), \quad V_d/G \dashrightarrow \mathcal{F}(\Gamma_{2d-9}),$$

where $\mathcal{F}(\Gamma_n)$ is the modular variety associated to Γ_n . One has $\Gamma_n = \widetilde{\mathrm{O}}(L_n)$ for $1 \leq n \leq 6$, and for $n = 7, 8$ one has $\Gamma_n \supset \widetilde{\mathrm{O}}(L_n)$ with $\Gamma_n/\widetilde{\mathrm{O}}(L_n) \simeq \mathfrak{S}_{n-5}$ where \mathfrak{S}_N is the symmetric group on N letters.

When $d \leq 6$, we recover some results of Matsumoto-Sasaki-Yoshida [26]. They constructed a period map for U_6 , and then obtained lower-dimensional period maps by degeneration. The novel part of Theorem 1.2 is the construction of the period maps for $d = 7, 8$ points. Also our period maps for $d \leq 6$ are derived from the ones for $d = 7, 8$, and are not identical to the ones of [26]. It is a future task to study the whole boundary behavior of the period maps.

Kondō, Dolgachev, and van Geemen [23], [10], [24] described the spaces U_d/G for $5 \leq d \leq 7$ as ball quotients. It is also known [11] that U_7/G can be described as a Siegel modular variety. Thus those spaces U_d/G admit (birationally) the structure of an arithmetic quotient in more than one way: after suitable compactifications, they may provide examples of “Janus-like” varieties (cf. [17]). In view of the relation with the moduli of del Pezzo surfaces, it would also be interesting to study the Weyl group action on $\mathcal{F}(\Gamma_{2d-8})$ induced by the period map.

The rest of the paper is structured as follows. In Section 2 we review the necessary facts concerning lattices, modular varieties, and invariant theory. In Section 3 we gather basic results on 2-elementary $K3$ surfaces with particular attention to the relation with singular sextic curves. The proof of Theorem 1.1 will be developed from Section 4 to Section 9. Theorem 1.2 will be proved in Sections 7, 8, and 9. In Section 10 we deduce the unirationality of the moduli spaces of Borcea-Voisin threefolds as a consequence of Theorem 1.1. In the Appendix A written by Yoshikawa, the approach by modular forms is presented.

Otherwise stated, we work in the category of algebraic varieties over \mathbb{C} .

2. PRELIMINARIES

2.1. Lattices. Let L be a *lattice*, i.e., a free \mathbb{Z} -module of finite rank endowed with a non-degenerate integral symmetric bilinear form (\cdot, \cdot) . The orthogonal group of L is denoted by $O(L)$. For an integer $n \neq 0$, $L(n)$ denotes the scaled lattice $(L, n(\cdot, \cdot))$. The lattice L is *even* if $(l, l) \in 2\mathbb{Z}$ for all $l \in L$, and *odd* otherwise. The dual lattice $L^\vee = \text{Hom}(L, \mathbb{Z})$ of L is canonically embedded in $L \otimes \mathbb{Q}$ and contains L . On the finite abelian group $D_L = L^\vee/L$ we have the \mathbb{Q}/\mathbb{Z} -valued bilinear form b_L defined by $b_L(x + L, y + L) = (x, y) + \mathbb{Z}$. We denote by $\widetilde{O}(L) \subset O(L)$ the group of isometries of L which act trivially on D_L . When L is even, b_L is induced by the quadratic form $q_L: D_L \rightarrow \mathbb{Q}/2\mathbb{Z}$, $q_L(x + L) = (x, x) + 2\mathbb{Z}$, which is called the *discriminant form* of L . We denote by $r_L: O(L) \rightarrow O(D_L, q_L)$ the natural homomorphism.

Proposition 2.1 ([30]). *Let Λ be an even unimodular lattice and L be a primitive sublattice of Λ with the orthogonal complement M . Then one has a natural isometry $\lambda: (D_L, q_L) \simeq (D_M, -q_M)$ defined by the relation $x + \lambda(x) \in \Lambda$, $x \in D_L$. For two isometries $\gamma_L \in O(L)$ and $\gamma_M \in O(M)$, the isometry $\gamma_L \oplus \gamma_M$ of $L \oplus M$ extends to that of Λ if and only if $r_L(\gamma_L) = \lambda^{-1} \circ r_M(\gamma_M) \circ \lambda$.*

A lattice L is called *2-elementary* if D_L is 2-elementary, i.e., $D_L \simeq (\mathbb{Z}/2\mathbb{Z})^a$ for some $a \geq 0$. The *main invariant* of an even 2-elementary lattice L is the quadruplet (r_+, r_-, a, δ) where (r_+, r_-) is the signature of L , a is the length of D_L as above, and δ is defined by $\delta = 0$ if $q_L(D_L) \subset \mathbb{Z}/2\mathbb{Z}$ and $\delta = 1$ otherwise. By Nikulin [30], the isometry class of L is uniquely determined by the main invariant if L is indefinite. When L is hyperbolic, we also call the triplet $(1 + r_-, a, \delta)$ the main invariant of L . In this paper we often use the following 2-elementary lattices with basis:

$$(2.1) \quad M_n = \langle 2 \rangle \oplus \langle -2 \rangle^{n-1} = \langle h, e_1, \dots, e_{n-1} \rangle,$$

$$(2.2) \quad U(2) = \langle u, v \rangle,$$

where $\{h, e_1, \dots, e_{n-1}\}$ are orthogonal basis with $(h, h) = 2$ and $(e_i, e_i) = -2$, and $\{u, v\}$ are basis with $(u, u) = (v, v) = 0$ and $(u, v) = 2$. Let

$$(2.3) \quad \Lambda_{K3} = U^3 \oplus E_8^2$$

be the even unimodular lattice of signature $(3, 19)$ where U is the hyperbolic plane (the scaling of $U(2)$ by $\frac{1}{2}$) and E_8 is the rank 8 even negative-definitive unimodular lattice. The following assertion is due to Nikulin.

Proposition 2.2 ([30], [31]). *Let L be an even hyperbolic 2-elementary lattice. If a primitive embedding $L \hookrightarrow \Lambda_{K3}$ exists, then it is unique up to the action of $O(\Lambda_{K3})$.*

2.2. Orthogonal modular varieties. Let L be a lattice of signature $(2, r_-)$ and let $\Gamma \subset O(L)$ be a finite-index subgroup. The group Γ acts properly discontinuously on the complex manifold

$$\Omega_L = \{ \mathbb{C}\omega \in \mathbb{P}(L \otimes \mathbb{C}) \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0 \}.$$

The domain Ω_L has two connected components, say Ω_L^+ and Ω_L^- . We denote by Γ^+ the group of those isometries in Γ which preserve Ω_L^+ . The quotient space

$$(2.4) \quad \mathcal{F}_L(\Gamma^+) = \Gamma^+ \backslash \Omega_L^+$$

is a normal quasi-projective variety of dimension r_- , by [2], called the modular variety associated to Γ^+ . When the lattice L is understood from the context, we abbreviate $\mathcal{F}_L(\Gamma^+)$ as $\mathcal{F}(\Gamma^+)$.

Proposition 2.3. *Let L be a finite-index sublattice of a lattice M of signature $(2, r_-)$. Then there exists a finite surjective morphism $\mathcal{F}(\widetilde{\mathcal{O}}(L)^+) \rightarrow \mathcal{F}(\widetilde{\mathcal{O}}(M)^+)$.*

Proof. We have the sequence $L \subset M \subset M^\vee \subset L^\vee$ of inclusions. If we regard the finite groups $G_1 = M/L$ and $G_2 = M^\vee/L$ as subgroups of D_L , then we have $G_2 = \{x \in D_L, b_L(x, G_1) \equiv 0\}$ and the induced bilinear form $(G_2/G_1, b_L)$ is canonically isometric to (D_M, b_M) . Since the isometries in $\widetilde{\mathcal{O}}(L)$ act trivially on both G_1 and G_2 , they preserve the overlattice M of L , and as isometries of M act trivially on D_M . Thus we have a finite-index embedding $\widetilde{\mathcal{O}}(L) \hookrightarrow \widetilde{\mathcal{O}}(M)$ of groups. Via the natural identification $\Omega_L = \Omega_M \subset \mathbb{P}(L \otimes \mathbb{C}) = \mathbb{P}(M \otimes \mathbb{C})$, this embedding induces a finite morphism $\mathcal{F}(\widetilde{\mathcal{O}}(L)^+) \rightarrow \mathcal{F}(\widetilde{\mathcal{O}}(M)^+)$. \square

The following proposition was used by Kondō [21] to prove the rationality of the moduli space of Enriques surfaces.

Proposition 2.4. *Let L be an even 2-elementary lattice of signature $(2, r_-)$. Then the lattice $M = L^\vee(2)$ is 2-elementary and we have $\mathcal{F}(\mathcal{O}(L)^+) \simeq \mathcal{F}(\mathcal{O}(M)^+)$.*

Proof. Since $L(2) \subset M \subset \frac{1}{2}L(2) = M^\vee$, we see that M is 2-elementary. We have the coincidence $\mathcal{O}(L) = \mathcal{O}(L^\vee)$ in $\mathcal{O}(L \otimes \mathbb{Q})$ because of the double dual relation $L^{\vee\vee} = L$. Thus we have $\mathcal{F}_L(\mathcal{O}(L)^+) \simeq \mathcal{F}_{L^\vee}(\mathcal{O}(L^\vee)^+) \simeq \mathcal{F}_M(\mathcal{O}(M)^+)$. \square

2.3. Geometric Invariant Theory. We review some facts from Geometric Invariant Theory. Throughout this section let X be a variety acted on by a reductive algebraic group G . A G -invariant morphism $\pi: X \rightarrow Y$ to a variety Y is a *geometric quotient* of X by G if (i) π is surjective, (ii) $\mathcal{O}_Y \simeq (\pi_* \mathcal{O}_X)^G$, (iii) a subset $U \subset Y$ is open if $\pi^{-1}(U) \subset X$ is open, and (iv) the fibers of π are the G -orbits. We sometimes denote $Y = X/G$ and omit π . A geometric quotient $\pi: X \rightarrow Y$ enjoys the following universality: for every G -invariant morphism $f: X \rightarrow Z$ there exists a unique morphism $g: Y \rightarrow Z$ with $g \circ \pi = f$. In particular, a geometric quotient, if it exists, is unique up to isomorphism.

Let L be an ample G -linearized line bundle on X . A point $x \in X$ is *stable* (with respect to L) if (i) the stabilizer G_x is a finite group, and (ii) there is a G -invariant section $s \in H^0(L^{\otimes n})^G$ for some $n > 0$ such that $s(x) \neq 0$ and that the action of G on $X_s = \{x' \in X, s(x') \neq 0\}$ is closed. The open set of stable points is denoted by $X^s(L)$.

Theorem 2.5 ([28]). *Let X, G, L be as above. Then a geometric quotient $X^s(L)/G$ of $X^s(L)$ exists and is a quasi-projective variety.*

Lemma 2.6. *Let $f: X \rightarrow Y$ be a G -equivariant finite morphism of G -varieties. Suppose we have an ample G -linearized line bundle L on Y such that $Y = Y^s(L)$. Then we have $X = X^s(f^*L)$. In particular, we have a geometric quotient X/G .*

Proof. Note that f^*L is ample and naturally G -linearized. For every $x \in X$ the stabilizer G_x is a subgroup of $G_{f(x)}$ and hence is finite. For an invariant section $s \in H^0(Y, L^{\otimes n})^G$ with $s(f(x)) \neq 0$ and with closed G -action on Y_s , we have $f^*s(x) \neq 0$ and the G -action on $X_{f^*s} = f^{-1}(Y_s)$ is also closed. \square

We will apply the machinery of GIT to plane sextic curves ([32]), bidegree $(4, 4)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$ ([33]), and point sets in \mathbb{P}^2 ([28], [11]).

Definition 2.7. Let $C \subset S$ be a reduced curve on a smooth surface S . A singular point $p \in C$ is a *simple singularity* if (i) p is a double or triple point, and (ii) the strict transform of C in the blow-up of S at p does not have triple point over p .

See [3] II.8 for the A-D-E classification of the simple singularities. In this paper we will deal mainly with nodes (A_1 -points) and ordinary triple points (D_4 -points). In some literatures, the condition (ii) above is stated in the form “ C has no consecutive triple point” ([32]) or “ C has no infinitely near triple point” ([16]).

We consider the PGL_3 -action on the linear system $|O_{\mathbb{P}^2}(6)|$ of plane sextic curves, which is endowed with a natural linearized ample line bundle.

Proposition 2.8 (Shah [32]). *A reduced plane sextic is PGL_3 -stable if and only if it has only simple singularities.*

We also need a stability criterion for the $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ -action on the linear system $|O_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 4)|$ endowed with the naturally linearized $O(1)$.

Proposition 2.9 (Shah [33]). *Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a reduced curve of bidegree $(4, 4)$. If C has only nodes as singularities, then C is $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ -stable.*

Finally we consider the diagonal action of PGL_3 on the product $(\mathbb{P}^2)^d$. Let $U_d \subset (\mathbb{P}^2)^d$ be the open set of ordered points (p_1, \dots, p_d) such that no three of $\{p_i\}_{i=1}^d$ are collinear, and let $V_d \subset (\mathbb{P}^2)^d$ be the variety of ordered points (p_1, \dots, p_d) such that $\{p_1, p_2, p_3\}$ are collinear and no other three of $\{p_i\}_{i=1}^d$ are collinear.

Proposition 2.10 ([28], [11]). *For $d \geq 4$ (resp. $d \geq 5$) a geometric quotient U_d/PGL_3 (resp. V_d/PGL_3) exists and is a quasi-projective rational variety of dimension $2d - 8$ (resp. $2d - 9$).*

Proof. For the assertion for U_d , see [11] Chapter II. The variety V_d is contained in the stable locus with respect to the SL_3 -linearized line bundle $O_{\mathbb{P}^2}(1) \boxtimes \dots \boxtimes O_{\mathbb{P}^2}(1)$ so that a geometric quotient exists by Theorem 2.5. For $d \geq 7$ the rationality of V_d/PGL_3 follows from the birational equivalence $V_d/\mathrm{PGL}_3 \sim V_{d-4}$. The remaining V_5/PGL_3 and V_6/PGL_3 are also clearly rational. \square

3. 2-ELEMENTARY $K3$ SURFACES

3.1. Basic properties. We recall basic facts on 2-elementary $K3$ surfaces following [31] and [1]. Let (X, ι) be a 2-elementary $K3$ surface, i.e., a pair of a complex

$K3$ surface X and a non-symplectic involution ι on X . The surface X is always algebraic due to the presence of ι . The invariant and anti-invariant lattices

$$(3.1) \quad L_{\pm} = L_{\pm}(X, \iota) = \{l \in H^2(X, \mathbb{Z}), \iota^* l = \pm l\}$$

are even 2-elementary lattices of signature $(1, r-1)$ and $(2, 20-r)$ respectively, where r is the rank of L_+ . Note that L_- is the orthogonal complement of L_+ in $H^2(X, \mathbb{Z})$ and hence we have a natural isometry $(D_{L_+}, q_{L_+}) \simeq (D_{L_-}, -q_{L_-})$. The main invariant (r, a, δ) of L_+ is also called the main invariant of (X, ι) and may be calculated geometrically as follows.

Proposition 3.1 ([31]). *Let (X, ι) be a 2-elementary $K3$ surface of type (r, a, δ) . Let X^{ι} be the fixed locus of ι .*

- (i) *If $(r, a, \delta) = (10, 10, 0)$, then $X^{\iota} = \emptyset$.*
- (ii) *If $(r, a, \delta) = (10, 8, 0)$, then X^{ι} is a union of two elliptic curves.*
- (iii) *In other cases, X^{ι} is decomposed as $X^{\iota} = C^g \sqcup E_1 \sqcup \cdots \sqcup E_k$ where C^g is a genus g curve and E_1, \dots, E_k are (-2) -curves with*

$$(3.2) \quad g = 11 - \frac{r+a}{2}, \quad k = \frac{r-a}{2}.$$

One has $\delta = 0$ if and only if the class of X^{ι} is divisible by 2 in NS_X .

Let $f: X \rightarrow Y = X/\langle \iota \rangle$ be the quotient morphism and $B = f(X^{\iota})$ be the branch curve of f . If $X^{\iota} \neq \emptyset$, Y is a smooth rational surface and B is a smooth member of $|-2K_Y|$. Following [1], we call such a pair (Y, B) a *right DPN pair*. The 2-elementary $K3$ surface (X, ι) is recovered from (Y, B) as the double cover of Y branched over B . In this way, 2-elementary $K3$ surfaces with non-empty fixed locus are in canonical correspondence with right DPN pairs. The invariant (r, a) of (X, ι) can be read off from the topology of B by Proposition 3.1. We also have

$$(3.3) \quad r = \rho(Y).$$

For the parity δ , if $B = \sum_i B_i$ is the irreducible decomposition of B , then we have $\delta = 0$ if and only if the class $\sum_i (-1)^{n_i} [B_i]$ is divisible by 4 in NS_Y for some $n_i \in \{0, 1\}$. The lattice $L_+(X, \iota)$ may be obtained as follows.

Proposition 3.2. *Let (Y, B) be a right DPN pair and (X, ι) be the associated 2-elementary $K3$ surface with the quotient morphism $f: X \rightarrow Y$. Then the invariant lattice $L_+ = L_+(X, \iota)$ is generated by the sublattice f^*NS_Y and the classes of irreducible components of X^{ι} .*

Proof. Let $B = \sum_{i=0}^k B_i$ be the irreducible decomposition and let $C_i = f^{-1}(B_i)$. We have $X^{\iota} = \sum_{i=0}^k C_i$ and $C_i \sim \frac{1}{2}f^*B_i$. According to Kharlamov ([19] p.304), the relation $\sum_{i=0}^k C_i \sim -f^*K_Y$ is the only relation among $\{C_i\}_{i=0}^k$ in L_+/f^*NS_Y . Since the lattice $f^*NS_Y \simeq NS_Y(2)$ is of index $2^{\frac{1}{2}(r-a)} = 2^k$ in L_+ , this proves the assertion. \square

3.2. Right resolutions of plane sextics. We explain a relationship between 2-elementary $K3$ surfaces and plane sextics with only simple singularities. The topic is classical as it goes back to Horikawa [16] and Shah [32]. Here we develop the argument in more generality in the framework of Alexeev-Nikulin [1]. Recall from [1] that a *DPN pair* is a pair (Y, B) of a smooth rational surface Y and an anti-bicanonical curve $B \in |-2K_Y|$ with only simple singularities.

Definition 3.3. A *right resolution* of a DPN pair (Y_0, B_0) is a triplet (Y, B, π) such that (Y, B) is a right DPN pair and $\pi: Y \rightarrow Y_0$ is a birational morphism with $\pi(B) = B_0$. By abuse of terminology, we also call (Y, B, π) a right resolution of B_0 when Y_0 is obvious from the context.

Proposition 3.4 (cf. [1]). *A right resolution of a DPN pair (Y_0, B_0) exists and is unique up to isomorphism.*

Proof. Let $S \rightarrow Y_0$ be the double cover branched over B_0 . As B_0 has only simple singularities, S is a normal surface with only A-D-E singularities (corresponding to those of B_0) and with trivial canonical divisor. The minimal resolution X of S is a $K3$ surface, and the covering transformation of $S \rightarrow Y_0$ induces a non-symplectic involution ι on X . If (Y, B) is the right DPN pair associated to (X, ι) , then by the universality of the quotient $X \rightarrow Y$ we have a birational morphism $\pi: Y \rightarrow Y_0$ with $\pi(B) = B_0$. This proves the existence. For any other right resolution (Y', B', π') with the associated 2-elementary $K3$ surface (X', ι') , let $X' \rightarrow S' \rightarrow Y_0$ be the Stein factorization of the morphism $X' \rightarrow Y' \rightarrow Y_0$. Then $S' \rightarrow Y_0$ is a double cover branched over B_0 and thus is isomorphic to $S \rightarrow Y_0$. It follows that $X' \rightarrow Y_0$ is isomorphic to $X \rightarrow Y_0$ and we have $(Y, B, \pi) \simeq (Y', B', \pi')$. \square

In [1] right resolution is constructed explicitly as follows. Let

$$(3.4) \quad \cdots \xrightarrow{\pi_{i+1}} (Y_i, B_i) \xrightarrow{\pi_i} (Y_{i-1}, B_{i-1}) \xrightarrow{\pi_{i-1}} \cdots \xrightarrow{\pi_1} (Y_0, B_0)$$

be the successive blow-ups defined inductively by

$$(3.5) \quad Y_{i+1} = \text{bl}_{\Sigma_i} Y_i, \quad B_{i+1} = \widetilde{B}_i + \sum_{k=1}^N (m_k - 2) E_k,$$

where $\Sigma_i = \{p_k\}_{k=1}^N$ is the singular locus of B_i , \widetilde{B}_i is the strict transform of B_i , E_k is the (-1) -curve over p_k , and m_k is the multiplicity of B_i at p_k . Each (Y_i, B_i) is also a DPN pair. This process will terminate and we finally obtain a right DPN pair (Y, B) .

In this way, one can associate a 2-elementary $K3$ surface (X, ι) to a DPN pair (Y_0, B_0) by taking its right resolution (Y, B, π) . Composing π with the quotient map $X \rightarrow Y$, we have a natural generically two-to-one morphism $g: X \rightarrow Y_0$ branched over B_0 . In this paper we will deal only with the following simple situations.

Example 3.5. When B_0 has only nodes p_1, \dots, p_a as the singularities, then $E_i = g^{-1}(p_i)$ is a (-2) -curve on X , and each component of the fixed curve X^ι is mapped by g birationally onto a component of B_0 . By Proposition 3.2 the lattice $L_+(X, \iota)$ is generated by the sublattice $g^*NS_{Y_0} \simeq M_{\rho(Y_0)}$, the classes of the (-2) -curves E_1, \dots, E_a , and of the components of X^ι . In particular, we have $r = \rho(Y_0) + a$.

Example 3.6. As a slight generalization, suppose that $\text{Sing}(B_0)$ consists of nodes p_1, \dots, p_a and ordinary triple points q_1, \dots, q_d . Then the curve $g^{-1}(q_j)$ is decomposed as $g^{-1}(q_j) = G_j + \sum_{k=1}^3 E_{jk}$ such that G_j is a rational component of X^t , and E_{jk} are the (-2) -curves over the infinitely near points of q_j given by the branches of B_0 . We have $(G_j.E_{jk}) = 1$ and $(E_{jk}.E_{jk'}) = -2\delta_{kk'}$. Other components of X^t than G_1, \dots, G_d are mapped by g birationally onto the components of B_0 . The lattice $L_+(X, \iota)$ is generated by $g^*NS_{Y_0}$, the classes of the (-2) -curves $g^{-1}(p_i)$, E_{jk} , G_j , and of those components of X^t . In particular, we have $r = \rho(Y_0) + a + 4d$.

When $Y_0 = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, for which B_0 is a sextic or a bidegree $(4, 4)$ curve respectively, we have the following useful property.

Lemma 3.7. *Let (Y_0, B_0) be a DPN pair with Y_0 being either \mathbb{P}^2 or a smooth quadric in \mathbb{P}^3 . Let (X, ι) be the associated 2-elementary K3 surface with the natural projection $g : X \rightarrow Y_0$. Then the morphism $g : X \rightarrow Y_0 \subset \mathbb{P}^d$ can be identified with the morphism $\phi_H : X \rightarrow |H|^\vee$ associated to the bundle $H = g^*\mathcal{O}_{Y_0}(1)$.*

Proof. The bundle H is nef and big. Use the Riemann-Roch formula and the vanishing $h^i(H) = 0$ for $i > 0$ to see that $|H| = g^*|\mathcal{O}_{Y_0}(1)|$. \square

3.3. Classification and the moduli spaces. 2-elementary K3 surfaces were classified by Nikulin in terms of the main invariants.

Theorem 3.8 (Nikulin [31]). *The deformation type of a 2-elementary K3 surface (X, ι) is determined by the main invariant (r, a, δ) . All possible main invariants of 2-elementary K3 surfaces are shown on the following Figure 1 which is identical to the table in page 31 of [1].*

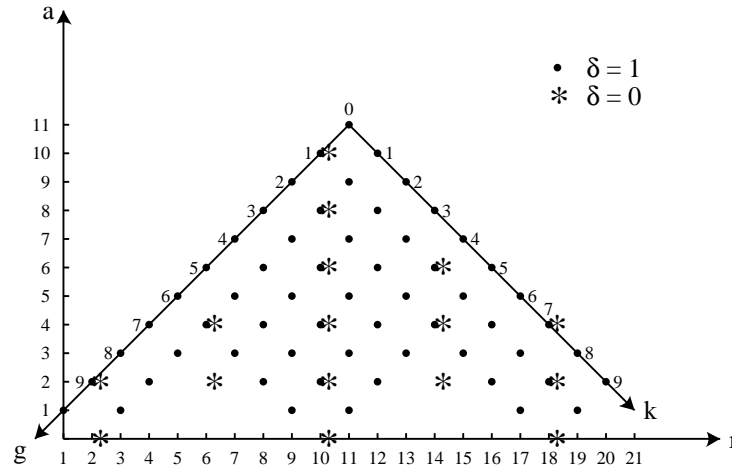


FIGURE 1. Geography of main invariants (r, a, δ)

A moduli space of 2-elementary K3 surfaces of main invariant (r, a, δ) is constructed as follows. We fix an even 2-elementary lattice L of main invariant

$(2, 20 - r, a, \delta)$, which is isometric to the anti-invariant lattice of every 2-elementary K3 surface of type (r, a, δ) . Let $\mathcal{F}(\mathrm{O}(L)^+) = \mathrm{O}(L)^+ \backslash \Omega_L^+$ be the modular variety associated to $\mathrm{O}(L)^+$. The divisor $\sum \delta^\perp \subset \Omega_L^+$, where δ are (-2) -vectors in L , is the inverse image of an algebraic divisor $D \subset \mathcal{F}(\mathrm{O}(L)^+)$. Let $\mathcal{M}_{(r,a,\delta)}$ be the variety

$$(3.6) \quad \mathcal{M}_{(r,a,\delta)} = \mathcal{F}(\mathrm{O}(L)^+) - D,$$

which is normal, irreducible, quasi-projective, and of dimension $20 - r$. For a 2-elementary K3 surface (X, ι) of type (r, a, δ) , we may choose an isometry $\Phi: L_-(X, \iota) \rightarrow L$ with $\Phi(H^{2,0}(X)) \in \Omega_L^+$. Then we define the period of (X, ι) by

$$(3.7) \quad \mathcal{P}(X, \iota) = [\Phi(H^{2,0}(X))] \in \mathcal{M}_{(r,a,\delta)},$$

which is independent of the choice of Φ .

Theorem 3.9 (Yoshikawa [36], [38]). *The variety $\mathcal{M}_{(r,a,\delta)}$ is a moduli space of 2-elementary K3 surfaces of type (r, a, δ) in the following sense.*

(i) *For a complex analytic family $(\mathfrak{X} \rightarrow U, \iota)$ of such 2-elementary K3 surfaces, the period map $\mathcal{P}: U \rightarrow \mathcal{M}_{(r,a,\delta)}, u \mapsto \mathcal{P}(\mathfrak{X}_u, \iota_u)$, is holomorphic. When the family is algebraic, \mathcal{P} is a morphism of algebraic varieties.*

(ii) *Via the period mapping, the points of $\mathcal{M}_{(r,a,\delta)}$ are in one-to-one correspondence with the isomorphism classes of 2-elementary K3 surfaces of type (r, a, δ) .*

3.4. The discriminant covers. Let L be the lattice used in the definition (3.6) and $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ be the modular variety

$$(3.8) \quad \widetilde{\mathcal{M}}_{(r,a,\delta)} = \mathcal{F}(\widetilde{\mathrm{O}}(L)^+),$$

which is a Galois cover of $\mathcal{F}(\mathrm{O}(L)^+)$ with the Galois group $\mathrm{O}(D_L, q_L)$. We call $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ the *discriminant cover* of $\mathcal{M}_{(r,a,\delta)}$. Since $\widetilde{\mathrm{O}}(L)^+ \neq \widetilde{\mathrm{O}}(L)$, we may identify $\widetilde{\mathcal{M}}_{(r,a,\delta)} = \widetilde{\mathrm{O}}(L) \backslash \Omega_L$. The next proposition is a key for our proof of Theorem 1.1.

Proposition 3.10. *Let (r, a, δ) and (r, a', δ') be main invariants of 2-elementary K3 surfaces. Assume that either (i) $\delta = 1, a > a'$, or (ii) $\delta = \delta', a > a'$. Then one has a finite surjective morphism $\varphi: \widetilde{\mathcal{M}}_{(r,a,\delta)} \rightarrow \widetilde{\mathcal{M}}_{(r,a',\delta')}$.*

Proof. Let L and L' be even 2-elementary lattices of main invariant $(2, 20 - r, a, \delta)$ and $(2, 20 - r, a', \delta')$ respectively. Calculating the discriminant form (D_L, q_L) explicitly, one can find an isotropic subgroup $G \subset D_L$ such that the 2-elementary quadratic form $(G^\perp/G, q_L)$ has the invariant (a', δ') . By the coincidence of main invariant, the overlattice of L defined by G is isometric to L' . Hence the assertion follows from Proposition 2.3. \square

The relationship between the modular varieties is as follows.

$$(3.9) \quad \begin{array}{ccc} \widetilde{\mathcal{M}}_{(r,a,\delta)} - H & \xrightarrow{\varphi} & \widetilde{\mathcal{M}}_{(r,a',\delta')} - H' \\ \downarrow & & \downarrow \\ \mathcal{M}_{(r,a,\delta)} & & \mathcal{M}_{(r,a',\delta')} \end{array}$$

Here H and H' are appropriate Heegner divisors.

Remark 3.11. When $a' = a - 2$, φ admits the following geometric interpretation. For an $\omega \in \widetilde{\mathcal{M}}_{(r,a,\delta)}$ let $(X, \iota) \in \mathcal{M}_{(r,a,\delta)}$ and $(X', \iota') \in \mathcal{M}_{(r,a',\delta')}$ be the 2-elementary $K3$ surfaces given by the images of ω and $\varphi(\omega)$ respectively. Then X is derived equivalent to the twisted $K3$ surface (X', α') for a Brauer element $\alpha' \in \text{Br}(X')$ of order ≤ 2 . Indeed, we have a Hodge embedding $T_X \hookrightarrow T_{X'}$ of the transcendental lattices of index ≤ 2 so that the twisted derived Torelli theorem [18] applies.

General points of $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ may be obtained as follows (cf. [9], [1]). We fix an even hyperbolic 2-elementary lattice L_+ of main invariant (r, a, δ) , a primitive embedding $L_+ \subset \Lambda_{K3}$, and an isometry $(L_+)^{\perp} \cap \Lambda_{K3} \rightarrow L$. Let $(X, \iota) \in \mathcal{M}_{(r,a,\delta)}$ and $j: L_+ \rightarrow L_+(X, \iota)$ be a given isometry. By Proposition 2.2 the isometry j can be extended to an isometry $\Phi: \Lambda_{K3} \rightarrow H^2(X, \mathbb{Z})$, which in turn induces the isometry $\Phi|_L: L \rightarrow L_-(X, \iota)$. By Proposition 2.1 the isometry $\Phi|_L$ is determined from j up to the action of $\widetilde{\text{O}}(L)$. Then we define the period of the lattice-marked 2-elementary $K3$ surface $((X, \iota), j)$ by

$$(3.10) \quad \widetilde{\mathcal{P}}((X, \iota), j) = [\Phi|_L^{-1}(H^{2,0}(X))] \in \widetilde{\mathcal{M}}_{(r,a,\delta)}.$$

If we define equivalence of two such objects $((X, \iota), j)$ and $((X', \iota'), j')$ by the existence of a Hodge isometry $\Psi: H^2(X, \mathbb{Z}) \rightarrow H^2(X', \mathbb{Z})$ with $j' = \Psi \circ j$, then via the period mapping $\widetilde{\mathcal{P}}$ the open set of $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ over $\mathcal{M}_{(r,a,\delta)}$ parametrizes the equivalence classes of such objects $((X, \iota), j)$. The assignment $((X, \iota), j) \mapsto (X, \iota)$ gives the projection $\widetilde{\mathcal{M}}_{(r,a,\delta)} \twoheadrightarrow \mathcal{M}_{(r,a,\delta)}$.

This interpretation of $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ using lattice-marked 2-elementary $K3$ surfaces is useful, but not so geometric. In the rest of this paper, using this interpretation intermediately, we will seek for more geometric interpretations for some of $\widetilde{\mathcal{M}}_{(r,a,\delta)}$.

Here is a general strategy. We define a space U parametrizing certain plane sextics B (or bidegree $(4, 4)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$) which are endowed with some labeling of their singularities and components. The 2-elementary $K3$ surface (X, ι) associated to the right resolution of B has main invariant (r, a, δ) . The point is that the labeling for B induces an isometry $j: L_+ \rightarrow L_+(X, \iota)$. Actually, an argument as in Examples 3.5 and 3.6 will suggest an appropriate definition of the reference lattice L_+ , and then j will be obtained naturally. Considering the period of $((X, \iota), j)$ as defined above, we obtain a morphism $p: U \rightarrow \widetilde{\mathcal{M}}_{(r,a,\delta)}$. We will prove that p descends to an open immersion $U/G \rightarrow \widetilde{\mathcal{M}}_{(r,a,\delta)}$ where $G = \text{PGL}_3$ (or $\text{PGL}_2 \times \text{PGL}_2$). This amounts to showing that $\dim(U/G) = 20 - r$ and that the p -fibers are G -orbits. The latter property is verified using the Torelli theorem and that the curve B with its labeling may be recovered from $((X, \iota), j)$ via Lemma 3.7.

In this way, some of $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ will be birationally identified with the moduli of certain curves with labeling. Such geometric interpretations vary according to $\widetilde{\mathcal{M}}_{(r,a,\delta)}$, and are out of single formulation. However, the processes by which we attach them to $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ are largely common, as suggested above. Then, in order to avoid repetition, we will discuss such processes in detail for only few cases (Section 4.1). For other cases, we omit some detail and refer to Section 4.1 as a model.

Now our geometric descriptions will imply that those $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ are often unirational. With the aid of Proposition 3.10, we will then obtain the following.

Theorem 3.12. *The discriminant covers $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ are unirational except possibly for $(r,a) = (10,10), (11,11), (12,10), (13,9)$.*

Sometimes our interpretations of $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ using sextics are translated into yet another geometric interpretations, such as configuration spaces of points in \mathbb{P}^2 .

4. THE CASE $r \leq 9$

In this section we prove that $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ are unirational for $r \leq 9$. We first prove in Section 4.1 the unirationality of $\widetilde{\mathcal{M}}_{(r,r,1)}$ with $r \leq 9$ using the Severi varieties of nodal plane sextics. These cases are model for the subsequent sections and hence discussed in detail. From Proposition 3.10 and Figure 1 follows the unirationality of $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ with $r \leq 9$ and $(r,a,\delta) \neq (2,2,0)$. In Section 4.2 we treat $\widetilde{\mathcal{M}}_{(2,2,0)}$.

4.1. $\widetilde{\mathcal{M}}_{(r,r,1)}$ and the Severi varieties of nodal sextics. For $r \leq 11$ let $V_{r-1} \subset |\mathcal{O}_{\mathbb{P}^2}(6)|$ be the variety of irreducible plane sextics with $r-1$ nodes and with no other singularity. The variety V_{r-1} , known as a *Severi variety*, is smooth, of dimension $28-r$, and irreducible ([15]). By endowing the sextics with markings of the nodes, we have the following \mathfrak{S}_{r-1} -cover of V_{r-1} :

$$(4.1) \quad \widetilde{V}_{r-1} = \{ (C, p_1, \dots, p_{r-1}) \in V_{r-1} \times (\mathbb{P}^2)^{r-1}, \text{Sing}(C) = \{p_i\}_{i=1}^{r-1} \}.$$

By Lemma 2.6 and Proposition 2.8 we have a geometric quotient $\widetilde{V}_{r-1}/\text{PGL}_3$.

Proposition 4.1. *For $r \leq 9$ the variety \widetilde{V}_{r-1} is rational. In particular, the quotient $\widetilde{V}_{r-1}/\text{PGL}_3$ is a unirational variety of dimension $20-r$.*

Proof. We consider the nodal map

$$(4.2) \quad \kappa: \widetilde{V}_{r-1} \rightarrow (\mathbb{P}^2)^{r-1}, \quad (C, p_1, \dots, p_{r-1}) \mapsto (p_1, \dots, p_{r-1}).$$

For a general $\mathbf{p} = (p_1, \dots, p_{r-1})$ the fiber $\kappa^{-1}(\mathbf{p})$ may be identified with an open set of $|-2K_Y|$ where Y is the blow-up of \mathbb{P}^2 at $\{p_i\}_{i=1}^{r-1}$. Since Y is a del Pezzo surface, we have $\dim |-2K_Y| \geq 3$ so that κ is dominant. As $\kappa^{-1}(\mathbf{p})$ is an open set of a linear subspace of $|\mathcal{O}_{\mathbb{P}^2}(6)|$, we see that \widetilde{V}_{r-1} is birationally equivalent to the projective bundle associated to a locally free sheaf on an open set of $(\mathbb{P}^2)^{r-1}$. \square

We shall construct a period map $\tilde{p}: \widetilde{V}_{r-1} \rightarrow \widetilde{\mathcal{M}}_{(r,r,1)}$ for $r \leq 11$. For a sextic with labeling $(C, \mathbf{p}) = (C, p_1, \dots, p_{r-1})$ in \widetilde{V}_{r-1} , let (X, ι) be the 2-elementary $K3$ surface associated to the right resolution of C , and $g: X \rightarrow \mathbb{P}^2$ be the natural projection branched over C . The quotient $X/\langle \iota \rangle$ is the blow-up of \mathbb{P}^2 at p_1, \dots, p_{r-1} . On X we have the line bundle $H = g^*\mathcal{O}_{\mathbb{P}^2}(1)$ and the (-2) -curves $E_i = g^{-1}(p_i)$. Let $M_r = \langle h, e_1, \dots, e_{r-1} \rangle$ be the lattice defined in (2.1). By Example 3.5, the classes of H and E_1, \dots, E_{r-1} define an isometry of lattices $j: M_r \rightarrow L_+(X, \iota)$ by $h \mapsto [H]$ and $e_i \mapsto [E_i]$. We thus associate a lattice-marked 2-elementary $K3$ surface $((X, \iota), j)$ to (C, \mathbf{p}) . Fixing a primitive embedding $M_r \hookrightarrow \Lambda_{K3}$ and considering the period of $((X, \iota), j)$ as defined in (3.10), we then obtain a point $\tilde{p}(C, \mathbf{p})$ in $\widetilde{\mathcal{M}}_{(r,r,1)}$.

Proposition 4.2. *Let $r \leq 11$. Two sextics with labeling $(C, \mathbf{p}), (C', \mathbf{p}') \in \widetilde{V}_{r-1}$ are PGL_3 -equivalent if and only if $\tilde{p}(C, \mathbf{p}) = \tilde{p}(C', \mathbf{p}')$.*

Proof. It suffices to prove the “if” part. Let X, j, H, \dots (resp. X', j', H', \dots) be the objects constructed from (C, \mathbf{p}) (resp. (C', \mathbf{p}')) as above. If $\tilde{p}(C, \mathbf{p}) = \tilde{p}(C', \mathbf{p}')$, we have a Hodge isometry $\Phi: H^2(X', \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$ with $j = \Phi \circ j'$. This equality means that $\Phi([H']) = [H]$ and $\Phi([E'_i]) = [E_i]$. Since Φ maps the ample class $4H' - \sum_{i=1}^{r-1} E'_i$ to the ample class $4H - \sum_{i=1}^{r-1} E_i$, by the strong Torelli theorem there exists an isomorphism $\varphi: X \rightarrow X'$ with $\varphi^* = \Phi$. Then we have $\varphi(E_i) = E'_i$ and $\varphi^*H' \simeq H$. By Lemma 3.7 we obtain an automorphism $\psi: \mathbb{P}^2 \rightarrow \mathbb{P}^2$ with $g' \circ \varphi = \psi \circ g$. Since $p_i = g(E_i)$ and $p'_i = g'(E'_i)$, we have $\psi(p_i) = p'_i$. Since C and C' are respectively the branches of g and g' , we also have $\psi(C) = C'$. \square

Theorem 4.3. *Let $r \leq 11$. The period map $\tilde{p}: \widetilde{V}_{r-1} \rightarrow \widetilde{\mathcal{M}}_{(r,r,1)}$ is a morphism of varieties and induces an open immersion $\widetilde{V}_{r-1}/\mathrm{PGL}_3 \rightarrow \widetilde{\mathcal{M}}_{(r,r,1)}$.*

Proof. We repeat the above construction for families. Let $\widetilde{C}_{r-1} \subset \widetilde{V}_{r-1} \times \mathbb{P}^2$ be the universal marked nodal sextic over \widetilde{V}_{r-1} (which may be obtained from the universal sextic over V_{r-1}). We have the sections $s_i: \widetilde{V}_{r-1} \rightarrow \widetilde{C}_{r-1}$ defined by $(C, \mathbf{p}) \mapsto ((C, \mathbf{p}), p_i)$ where $\mathbf{p} = (p_1, \dots, p_{r-1})$. There is an open set $\widetilde{V} \subset \widetilde{V}_{r-1}$ such that the divisor $\widetilde{C} = \widetilde{C}_{r-1}|_{\widetilde{V}}$ of $\widetilde{V} \times \mathbb{P}^2$ is linearly equivalent to $\pi_2^* \mathcal{O}_{\mathbb{P}^2}(6)$ where $\pi_2: \widetilde{V} \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is the projection. We denote $W_i = s_i(\widetilde{V})$. Let \mathcal{Y} be the blow-up of $\widetilde{V} \times \mathbb{P}^2$ along $\bigcup_{i=1}^{r-1} W_i$ and $\mathcal{D}_i \subset \mathcal{Y}$ be the exceptional divisor over W_i . Since the strict transform $\mathcal{B} \subset \mathcal{Y}$ of \widetilde{C} is a smooth divisor linearly equivalent to $\pi_2^* \mathcal{O}_{\mathbb{P}^2}(6) - 2 \sum_{i=1}^{r-1} \mathcal{D}_i$, we may take a double cover $f: \mathcal{X} \rightarrow \mathcal{Y}$ branched over \mathcal{B} . The natural projection $\pi: \mathcal{X} \rightarrow \widetilde{V}$ is a family of $K3$ surfaces. Let ι be the covering transformation of f and \mathcal{L}_+ be the local system $(R^2 \pi_* \mathbb{Z})^\iota$ over \widetilde{V} . Then the divisors $\{f^{-1}(\mathcal{D}_i)\}_i$ and the pullback of $\pi_2^* \mathcal{O}_{\mathbb{P}^2}(1)$ define a trivialization $\mathcal{L}_+ \rightarrow M_r \times \widetilde{V}$. This means that the monodromy group of the local system $\mathcal{L}_- = (\mathcal{L}_+)^{\perp} \cap R^2 \pi_* \mathbb{Z}$ is contained in $\widetilde{\mathrm{O}}(L_r)$ where $L_r = (M_r)^{\perp} \cap \Lambda_{K3}$. Considering the local system \mathcal{L}_- , we see that the period map $\tilde{p}|_{\widetilde{V}}: \widetilde{V} \rightarrow \widetilde{\mathcal{M}}_{(r,r,1)}$ is a locally liftable holomorphic map. By Borel’s extension theorem [7], $\tilde{p}|_{\widetilde{V}}$ is a morphism of algebraic varieties. This implies that \tilde{p} is a morphism of varieties. By the PGL_3 -invariance \tilde{p} induces a morphism $\widetilde{\mathcal{P}}: \widetilde{V}_{r-1}/\mathrm{PGL}_3 \rightarrow \widetilde{\mathcal{M}}_{(r,r,1)}$. Proposition 4.2 implies the injectivity of $\widetilde{\mathcal{P}}$. Then $\widetilde{\mathcal{P}}$ is dominant because we have $\dim(\widetilde{V}_{r-1}/\mathrm{PGL}_3) = 20 - r$ and $\widetilde{\mathcal{M}}_{(r,r,1)}$ is irreducible. Thus $\widetilde{\mathcal{P}}$ is an open immersion by the Zariski’s Main Theorem. \square

Corollary 4.4. *If $r \leq 9$ and $(r, a, \delta) \neq (2, 2, 0)$, then $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ is unirational.*

Proof. By Proposition 4.1 and Theorem 4.3, $\widetilde{\mathcal{M}}_{(r,r,1)}$ is unirational for $r \leq 9$. Then the assertion follows from Proposition 3.10 and Figure 1. \square

Remark 4.5. Morrison-Saito [27] constructed an open immersion $V_{r-1}/\mathrm{PGL}_3 \rightarrow \mathcal{F}(\Gamma_r)$ for a certain arithmetic group $\Gamma_r \subset \mathrm{O}(L_r)^+$. Our idea to relate $\widetilde{\mathcal{M}}_{(r,r,1)}$ with \widetilde{V}_{r-1} was inspired by their argument.

Remark 4.6. In fact, $\widetilde{V}_{r-1}/\mathrm{PGL}_3$ is rational when $2 \leq r \leq 9$. For $r \geq 5$ this may be seen by fixing first four nodes in general position. For $r \leq 4$ we need invariant-theoretic techniques. In the rest of the paper, one would find that several $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ are rational as well.

4.2. $\widetilde{\mathcal{M}}_{(2,2,0)}$ and bidegree (4, 4) curves. Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric embedded in \mathbb{P}^3 . The group $G = \mathrm{PGL}_2 \times \mathrm{PGL}_2$ acts naturally on Q . Let $U \subset |O_Q(4, 4)|$ be the open set of smooth bidegree (4, 4) curves. By Proposition 2.9 we have a geometric quotient U/G as an affine unirational variety of dimension 18.

For a curve $C \in U$ let (X, ι) be the 2-elementary $K3$ surface associated to the right DPN pair (Q, C) and $f: X \rightarrow Q$ be the quotient morphism. The lattice $L_+(X, \iota)$ is equal to f^*NS_Q by Proposition 3.2, and thus isometric to the lattice $U(2)$. In fact, using the basis $\{u, v\}$ of $U(2)$ defined in (2.2), we have an isometry $j: U(2) \rightarrow L_+(X, \iota)$ by $u \mapsto [f^*O_Q(1, 0)]$ and $v \mapsto [f^*O_Q(0, 1)]$. Here it is important to distinguish the two rulings of Q . In this way, we obtain a lattice-marked 2-elementary $K3$ surface $((X, \iota), j)$ from C . We then obtain a point $\tilde{p}(C)$ in $\widetilde{\mathcal{M}}_{(2,2,0)}$ as the period of $((X, \iota), j)$ as before.

In this construction, one may recover the morphism $f: X \rightarrow Q$ (and hence its branch C) from the class $j(u + v)$ by Lemma 3.7. By using f , the two rulings $|O_Q(1, 0)|, |O_Q(0, 1)|$ of Q may be respectively recovered from the elliptic fibrations on X given by the classes $j(u), j(v)$.

Theorem 4.7. *The period map $\tilde{p}: U \rightarrow \widetilde{\mathcal{M}}_{(2,2,0)}$ is a morphism of varieties and induces an open immersion $U/G \rightarrow \widetilde{\mathcal{M}}_{(2,2,0)}$. In particular, $\widetilde{\mathcal{M}}_{(2,2,0)}$ is unirational.*

Proof. Basically one may apply a similar argument as for Proposition 4.2 and Theorem 4.3. In the present case, one should note that G is the group of automorphisms of Q preserving the two rulings respectively. This ensures the G -invariance of \tilde{p} for its definition involves the distinction of the two rulings. The recovery of the morphisms f , the curves C , and the two rulings of Q as explained above implies the injectivity of the induced morphism $U/G \rightarrow \widetilde{\mathcal{M}}_{(2,2,0)}$. Here one may apply the strong Torelli theorem by using the ample classes $j(u + v)$. \square

5. THE CASE $r = 10$

In this section we prove that $\mathcal{M}_{(10,a,\delta)}$ are unirational. Kondō [21] proved the rationality of $\mathcal{M}_{(10,10,0)}$, the moduli of Enriques surfaces, and of $\mathcal{M}_{(10,2,0)}$. We study the remaining moduli spaces. In Sections 5.1 and 5.2 we prove the unirationality of $\widetilde{\mathcal{M}}_{(10,8,0)}$ and $\widetilde{\mathcal{M}}_{(10,8,1)}$ respectively, which implies that $\widetilde{\mathcal{M}}_{(10,a,\delta)}$ are unirational for $a \leq 8$. The unirationality of $\mathcal{M}_{(10,10,1)}$ is proved in Section 5.3.

5.1. $\widetilde{\mathcal{M}}_{(10,8,0)}$ and cubic pairs. Let $U \subset |O_{\mathbb{P}^2}(6)| \times (\mathbb{P}^2)^8$ be the space of pointed sextics $(C_1 + C_2, \mathbf{p}) = (C_1 + C_2, p_1, \dots, p_8)$ such that C_1 and C_2 are smooth cubics transverse to each other and that p_1, \dots, p_8 are distinct points contained in $C_1 \cap C_2$. The variety U is unirational. Indeed, if we denote by $V \subset |O_{\mathbb{P}^2}(3)| \times (\mathbb{P}^2)^8$ the locus of (C, p_1, \dots, p_8) such that $\{p_i\}_{i=1}^8 \subset C$, then U is dominated by the fiber product $V \times_{(\mathbb{P}^2)^8} V$. As the projection $V \rightarrow (\mathbb{P}^2)^8$ is dominant with a general fiber being

a line in $|\mathcal{O}_{\mathbb{P}^2}(3)|$, the variety $V \times_{(\mathbb{P}^2)^8} V$ is rational, and so U is unirational. By Proposition 2.8 and Lemma 2.6, the natural projection $U \rightarrow |\mathcal{O}_{\mathbb{P}^2}(6)|$ shows that we have a geometric quotient U/PGL_3 as a unirational variety of dimension 10.

For a pointed sextic $(C_1 + C_2, \mathbf{p}) \in U$ we denote by p_9 the ninth intersection point of C_1 and C_2 . This gives a complete labeling of the nodes of $C_1 + C_2$. Let (X, ι) be the 2-elementary $K3$ surface associated to $C_1 + C_2$ and $g: X \rightarrow \mathbb{P}^2$ be the natural projection branched over $C_1 + C_2$. The quotient $X/\langle \iota \rangle$ is the blow-up of \mathbb{P}^2 at p_1, \dots, p_9 , and is a rational elliptic surface. We have the decomposition $X^\iota = F_1 + F_2$ such that $g(F_i) = C_i$. By Example 3.5, the lattice $L_+(X, \iota)$ is generated by the classes of the bundle $H = g^* \mathcal{O}_{\mathbb{P}^2}(1)$, the (-2) -curves $E_i = g^{-1}(p_i)$ for $i \leq 9$, and the elliptic curves $F_1 \sim F_2$. This suggests to define a reference lattice L_+ as follows. Let $M_{10} = \langle h, e_1, \dots, e_9 \rangle$ be the lattice defined in (2.1) and $v \in M_{10}^\vee$ be the vector defined by $2v = 3h - \sum_{i=1}^9 e_i$. The even overlattice $L_+ = \langle M_{10}, v \rangle$ is 2-elementary of main invariant $(10, 8, 0)$. Then we have a natural isometry $j: L_+ \rightarrow L_+(X, \iota)$ by sending $h \mapsto [H]$, $e_i \mapsto [E_i]$, and $v \mapsto [F_j]$. Therefore we obtain a point $\tilde{p}(C_1 + C_2, \mathbf{p})$ in $\widetilde{\mathcal{M}}_{(10,8,0)}$ as the period of $((X, \iota), j)$ as before.

As in Section 4.1, one may recover the morphism $g: X \rightarrow \mathbb{P}^2$ from the class $j(h)$ by Lemma 3.7, the points $p_i = g(E_i)$ from the classes $j(e_i)$, and the sextic $C_1 + C_2$ from g as the branch locus. Also one has the ample class $j(h + v)$ on X defined in terms of j . Hence one may proceed as Section 4.1 to see the following.

Theorem 5.1. *The period map $\tilde{p}: U \rightarrow \widetilde{\mathcal{M}}_{(10,8,0)}$ is a morphism of varieties and descends to an open immersion $U/\mathrm{PGL}_3 \rightarrow \widetilde{\mathcal{M}}_{(10,8,0)}$.*

Corollary 5.2. *If $a \leq 8$, then $\widetilde{\mathcal{M}}_{(10,a,0)}$ is unirational.*

5.2. $\widetilde{\mathcal{M}}_{(10,8,1)}$ and bidegree $(3, 2)$ curves. Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric in \mathbb{P}^3 and let $G = \mathrm{PGL}_2 \times \mathrm{PGL}_2$. Let $U \subset |\mathcal{O}_Q(4, 4)| \times Q^8$ be the variety of pointed bidegree $(4, 4)$ curves $(C + D, \mathbf{p}) = (C + D, p_1, \dots, p_8)$ such that (i) C is smooth of bidegree $(3, 2)$, (ii) D is smooth of bidegree $(1, 2)$ and transverse to C , and (iii) $C \cap D = \{p_1, \dots, p_8\}$. The space U is an \mathfrak{S}_8 -cover of an open set of $|\mathcal{O}_Q(3, 2)| \times |\mathcal{O}_Q(1, 2)|$. By Proposition 2.9 and Lemma 2.6, we have a geometric quotient U/G as a 10-dimensional variety.

Lemma 5.3. *The variety U is rational.*

Proof. Let V be the linear system $|\mathcal{O}_Q(1, 2)|$ and $\mathcal{X} \subset V \times Q$ be the universal curve over V . The projection $\pi_1: \mathcal{X} \rightarrow V$ is birationally equivalent to the natural projection $\mathbb{P}^1 \times V \rightarrow V$ for bidegree $(0, 1)$ curves on Q give sections of π_1 . This implies that the fiber product $\mathcal{Y} = \mathcal{X} \times_V \mathcal{X} \cdots \times_V \mathcal{X}$ (8 times) is rational. We have a morphism $\pi_2: U \rightarrow \mathcal{Y}$ defined by $(C + D, \mathbf{p}) \mapsto (D, \mathbf{p})$. Then π_2 is dominant. Indeed, for every smooth $D \in V$ the restriction map $|\mathcal{O}_Q(3, 2)| \dashrightarrow |\mathcal{O}_D(8)|$ is dominant by the vanishing of $H^1(\mathcal{O}_Q(2, 0))$. Since a general π_2 -fiber is an open set of a linear subspace of $|\mathcal{O}_Q(3, 2)|$, this proves the rationality of U . \square

For a curve with labeling $(C + D, \mathbf{p}) \in U$, let (X, ι) be the 2-elementary $K3$ surface associated to the DPN pair $(Q, C + D)$ and $g: X \rightarrow Q$ be the natural projection

branched over $C + D$. The fixed curve X^t is decomposed as $X^t = F_1 + F_2$ such that $g(F_1) = C$ and $g(F_2) = D$. In this case, a reference lattice L_+ should be defined as follows. Let M be the lattice $U(2) \oplus \langle -2 \rangle^8 = \langle u, v, e_1, \dots, e_8 \rangle$ where $\{u, v\}$ is the basis of $U(2)$ defined in (2.2) and $\{e_1, \dots, e_8\}$ is a natural basis of $\langle -2 \rangle^8$. Let $f_1, f_2 \in M^\vee$ be the vectors defined by $2f_1 = 3u + 2v - \sum_{i=1}^8 e_i$ and $2f_2 = u + 2v - \sum_{i=1}^8 e_i$. The overlattice $L_+ = \langle M, f_1, f_2 \rangle$ is even and 2-elementary of main invariant $(10, 8, 1)$. Then, by Example 3.5, we have a natural isometry $j: L_+ \rightarrow L_+(X, \iota)$ by sending $u \mapsto [g^*O_Q(1, 0)]$, $v \mapsto [g^*O_Q(0, 1)]$, $e_i \mapsto [g^{-1}(p_i)]$, and $f_j \mapsto [F_j]$. In this way we associate to $(C + D, \mathbf{p})$ a lattice-marked 2-elementary K3 surface $((X, \iota), j)$, and hence a point $\tilde{p}(C + D, \mathbf{p})$ in $\widetilde{\mathcal{M}}_{(10, 8, 1)}$.

As in Section 4.2, the morphism $g: X \rightarrow Q$, the curve $C + D$, and the two rulings of Q are recovered from j . The points p_i are recovered from the classes $j(e_i)$. Therefore we have

Theorem 5.4. *The period map $\tilde{p}: U \rightarrow \widetilde{\mathcal{M}}_{(10, 8, 1)}$ is a morphism of varieties and descends to an open immersion $U/G \rightarrow \widetilde{\mathcal{M}}_{(10, 8, 1)}$.*

Corollary 5.5. *If $a \leq 8$, then $\widetilde{\mathcal{M}}_{(10, a, 1)}$ is unirational.*

5.3. The unirationality of $\mathcal{M}_{(10, 10, 1)}$. By Theorem 4.3, general members of $\mathcal{M}_{(10, 10, 1)}$ are obtained from *Halphen curves*, irreducible nine-nodal sextics. However, since the nodal map $\widetilde{V}_9 \rightarrow (\mathbb{P}^2)^9$ for Halphen curves is not dominant (see [8] p.389–p.391), our proof of Proposition 4.1 does not apply to \widetilde{V}_9 . Here we instead prove the unirationality of $\mathcal{M}_{(10, 10, 1)}$ using the description as a modular variety.

Theorem 5.6. *The moduli space $\mathcal{M}_{(10, 10, 1)}$ is unirational.*

Proof. Recall that $\mathcal{M}_{(10, 10, 1)}$ is an open set of the arithmetic quotient $\mathcal{F}(\mathcal{O}(L_1)^+)$ for the lattice $L_1 = U \oplus \langle 2 \rangle \oplus \langle -2 \rangle \oplus E_8(2)$. By Proposition 2.4 we have an isomorphism $\mathcal{F}(\mathcal{O}(L_1)^+) \simeq \mathcal{F}(\mathcal{O}(L_2)^+)$ for the odd lattice $L_2 = U(2) \oplus \langle 1 \rangle \oplus \langle -1 \rangle \oplus E_8$. Let L_3 be the lattice $U(2)^2 \oplus E_8$ and $\{u, v\}$ be the basis of its second summand $U(2)$ as defined in (2.2). Then L_2 is isometric to the overlattice $\langle L_3, \frac{1}{2}(u+v) \rangle$ of L_3 . Thus $\mathcal{F}(\widetilde{\mathcal{O}}(L_2)^+)$ is dominated by $\mathcal{F}(\widetilde{\mathcal{O}}(L_3)^+)$ by Proposition 2.3. The variety $\mathcal{F}(\widetilde{\mathcal{O}}(L_3)^+) = \widetilde{\mathcal{M}}_{(10, 4, 0)}$ is unirational by Corollary 5.2. Hence $\mathcal{F}(\mathcal{O}(L_1)^+)$ is unirational. \square

Remark 5.7. Alternatively, considering morphisms to \mathbb{P}^2 of genus 1 and degree 6, one can prove that V_9 is unirational using e.g., the relative Poincaré bundle for a rational elliptic surface with a section.

6. THE CASE $r = 11$

In this section we prove that $\mathcal{M}_{(11, 11, 1)}$ is unirational (Section 6.1) and that the covers $\widetilde{\mathcal{M}}_{(11, a, \delta)}$ are unirational for $a \leq 9$ (Section 6.2).

6.1. $\mathcal{M}_{(11, 11, 1)}$ and Coble curves. Let \widetilde{V}_{10} be the variety defined in (4.1). By Theorem 4.3 we have an open immersion $\widetilde{V}_{10}/\mathrm{PGL}_3 \rightarrow \widetilde{\mathcal{M}}_{(11, 11, 1)}$ and hence a dominant morphism $\mathcal{P}: \widetilde{V}_{10}/\mathrm{PGL}_3 \rightarrow \mathcal{M}_{(11, 11, 1)}$. Clearly, \mathcal{P} descends to a morphism $V_{10}/\mathrm{PGL}_3 \rightarrow \mathcal{M}_{(11, 11, 1)}$. The Severi variety V_{10} is dense in the variety of rational

plane sextics (cf. [15]). As the latter is dominated by the variety of morphisms $\mathbb{P}^1 \rightarrow \mathbb{P}^2$ of degree 6, which is obviously rational, we have the following.

Theorem 6.1. *The moduli space $\mathcal{M}_{(11,11,1)}$ is unirational.*

6.2. $\widetilde{\mathcal{M}}_{(11,9,1)}$ **and degenerated cubic pairs.** Let $U \subset |\mathcal{O}_{\mathbb{P}^2}(6)| \times (\mathbb{P}^2)^8$ be the variety of pointed sextics $(C_1 + C_2, \mathbf{p}) = (C_1 + C_2, p_1, \dots, p_8)$ such that C_1 is a smooth cubic, that C_2 is an irreducible cubic with a node and transverse to C_1 , and that p_1, \dots, p_8 are distinct points contained in $C_1 \cap C_2$. Letting p_9 be the remaining intersection point of C_1 and C_2 , and p_{10} be the node of C_2 , we have the complete labeling (p_1, \dots, p_{10}) of the nodes of $C_1 + C_2$. As in Section 5.1, we have a geometric quotient U/PGL_3 as a 9-dimensional variety.

Lemma 6.2. *The variety U is unirational.*

Proof. Let V denote the variety of irreducible cubics with nodes and $C \subset V \times \mathbb{P}^2$ be the universal curve over V . Let $X = C \times_V C \cdots \times_V C$ (8 times). We have a morphism $\pi: U \rightarrow X$ defined by $(C_1 + C_2, \mathbf{p}) \mapsto (C_2, \mathbf{p})$. A general π -fiber is an open set of a line in $|\mathcal{O}_{\mathbb{P}^2}(3)|$, namely the linear system $|-K_Y|$ for the blow-up Y of \mathbb{P}^2 at $\{p_i\}_{i=1}^8$. Therefore U is birational to $X \times \mathbb{P}^1$. Take a nodal cubic $[C] \in V$. Since $\mathrm{PGL}_3 \cdot [C] = V$, we have $\mathrm{PGL}_3 \cdot (C)^8 = X$ and hence X is unirational. \square

For a pointed sextic $(C_1 + C_2, \mathbf{p}) \in U$, the 2-elementary $K3$ surface (X, ι) associated to $C_1 + C_2$ has main invariant $(11, 9, 1)$. As before, the above labeling of the nodes induces a natural isometry $j: L_+ \rightarrow L_+(X, \iota)$ from a reference lattice L_+ , and this defines a morphism $\tilde{p}: U \rightarrow \widetilde{\mathcal{M}}_{(11,9,1)}$. Then we see the following.

Theorem 6.3. *The period map \tilde{p} descends to an open immersion $U/\mathrm{PGL}_3 \rightarrow \widetilde{\mathcal{M}}_{(11,9,1)}$.*

Corollary 6.4. *For $a \leq 9$ the covers $\widetilde{\mathcal{M}}_{(11,a,\delta)}$ are unirational.*

7. THE CASE $r = 12$

In this section we study the case $r = 12$. In Section 7.1 we construct a birational map from the configuration space of eight general points in \mathbb{P}^2 to a certain cover of $\mathcal{M}_{(12,10,1)}$, which in particular implies that $\mathcal{M}_{(12,10,1)}$ is unirational. In Section 7.2 we prove that the covers $\widetilde{\mathcal{M}}_{(12,a,\delta)}$ for $a \leq 8$ are unirational.

7.1. $\mathcal{M}_{(12,10,1)}$ **and eight general points in \mathbb{P}^2 .** We begin by preparing lattices and an arithmetic group. Let $M_{12} = \langle h, e_1, \dots, e_{11} \rangle$ be the lattice defined in (2.1). Let $f_1, f_2 \in M_{12}^\vee$ be the vectors defined by $2f_i = 3h - 2e_i - \sum_{j=3}^{11} e_j$, $i = 1, 2$. Then the overlattice $L_+ = \langle M_{12}, f_1, f_2 \rangle$ is even and 2-elementary of main invariant $(12, 10, 1)$. We fix a primitive embedding $L_+ \subset \Lambda_{K3}$, which exists by Table 1, and set $L_- = (L_+)^\perp \cap \Lambda_{K3}$. The lattice L_- is isometric to $\langle 2 \rangle^2 \oplus \langle -2 \rangle^8$. We let the symmetric group \mathfrak{S}_3 act on the set $\{e_9, e_{10}, e_{11}\}$ by permutation, and on the set $\{h, e_1, \dots, e_8\}$ trivially. This defines an action $i: \mathfrak{S}_3 \rightarrow \mathrm{O}(L_+)$ of \mathfrak{S}_3 on the lattice L_+ . Let $r_\pm: \mathrm{O}(L_\pm) \rightarrow \mathrm{O}(D_{L_\pm})$ be the natural homomorphisms and $\lambda: \mathrm{O}(D_{L_+}) \simeq \mathrm{O}(D_{L_-})$ be the isomorphism induced by the relation $L_- = (L_+)^\perp$. Then we define

a subgroup of $O(L_-)$ by $\Gamma = r_-^{-1}(\lambda \circ r_+(i(\mathfrak{S}_3)))$. By Proposition 2.1 an isometry γ of L_- is contained in Γ if and only if there exists a $\sigma \in \mathfrak{S}_3$ such that $i(\sigma) \oplus \gamma$ extends to an isometry of Λ_{K3} . We have $\widetilde{O}(L_-) \subset \Gamma$ with $\Gamma^+/\widetilde{O}(L_-)^+ \simeq \mathfrak{S}_3$. Hence the modular variety $\mathcal{F}_{L_-}(\Gamma^+)$ is a quotient of $\widetilde{\mathcal{M}}_{(12,10,1)}$ by \mathfrak{S}_3 . The moduli space $\mathcal{M}_{(12,10,1)}$ is dominated by $\mathcal{F}_{L_-}(\Gamma^+)$.

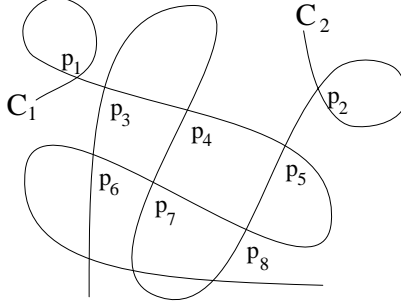


FIGURE 2. Sextic curve for $(r, a, \delta) = (12, 10, 1)$

We shall define a parameter space. First we note that for seven general points q_1, \dots, q_7 in \mathbb{P}^2 there uniquely exists an irreducible nodal cubic C passing q_1, \dots, q_7 with $\text{Sing}(C) = q_1$. This may be seen by an intersection calculation and a dimension counting. More constructively, the blow-up Y of \mathbb{P}^2 at q_1, \dots, q_7 is a quadric del Pezzo surface which has the Geisser involution ι . If $E \subset Y$ is the (-1) -curve over q_1 , then the image of the curve $\iota(E)$ in \mathbb{P}^2 is the desired cubic. Now let $U \subset (\mathbb{P}^2)^8$ be the open set of eight distinct points $\mathbf{p} = (p_1, \dots, p_8)$ such that there exist irreducible nodal cubics C_1, C_2 which pass p_3, \dots, p_8 with $\text{Sing}(C_i) = p_i$ and which are transverse to each other. The finite morphism $U \rightarrow |\mathcal{O}_{\mathbb{P}^2}(6)|$, $\mathbf{p} \mapsto C_1 + C_2$, shows that we have a geometric quotient U/PGL_3 as an 8-dimensional variety, which is rational by Proposition 2.10.

For a $\mathbf{p} = (p_1, \dots, p_8) \in U$ the associated sextic $C_1 + C_2$ is endowed with the partial labeling (p_1, \dots, p_8) of its nodes. The remaining three nodes $\mathcal{S} = C_1 \cap C_2 \setminus \{p_i\}_{i=3}^8$ are not marked. We temporarily choose a bijection $\mathcal{S} \simeq \{9, 10, 11\}$ and accordingly denote $\mathcal{S} = \{p_9, p_{10}, p_{11}\}$. Then let (X, ι) be the 2-elementary $K3$ surface associated to $C_1 + C_2$. If $g: X \rightarrow \mathbb{P}^2$ is the natural projection branched over $C_1 + C_2$, we have an isometry $j: L_+ \rightarrow L_+(X, \iota)$ defined by $h \mapsto [g^* \mathcal{O}_{\mathbb{P}^2}(1)]$, $e_i \mapsto [g^{-1}(p_i)]$ for $i \leq 11$, and $f_j \mapsto [F_j]$ where F_j is the component of X' with $g(F_j) = C_j$. Then the period of $((X, \iota), j)$ is determined as a point in $\widetilde{\mathcal{M}}_{(12,10,1)}$. We consider the image of that point in $\mathcal{F}_{L_-}(\Gamma^+)$, and denote it by $\mathcal{P}(\mathbf{p}) \in \mathcal{F}_{L_-}(\Gamma^+)$.

Theorem 7.1. *The map $\mathcal{P}: U \rightarrow \mathcal{F}_{L_-}(\Gamma^+)$ is well-defined. It is a morphism of varieties and induces an open immersion $U/\text{PGL}_3 \rightarrow \mathcal{F}_{L_-}(\Gamma^+)$.*

Proof. For the first assertion it suffices to show that $\mathcal{P}(\mathbf{p})$ is independent of the choice of a labeling $\mathcal{S} = \{p_9, p_{10}, p_{11}\}$. For another labeling $\mathcal{S} = \{p'_9, p'_{10}, p'_{11}\}$ we

have $p_{\sigma(i)} = p'_i$ for a $\sigma \in \mathfrak{S}_3$, $9 \leq i \leq 11$. Then the isometry $j': L_+ \rightarrow L_+(X, \iota)$ associated to (p'_9, p'_{10}, p'_{11}) is given by $j' = j \circ i(\sigma)$. If $\Phi, \Phi': \Lambda_{K3} \rightarrow H^2(X, \mathbb{Z})$ are extensions of j and j' respectively, then $\Phi|_{L_-}$ is Γ -equivalent to $\Phi'|_{L_-}$.

The map \mathcal{P} is obviously PGL_3 -invariant. Conversely, suppose that $\mathcal{P}(\mathbf{p}) = \mathcal{P}(\mathbf{p}')$ for two $\mathbf{p}, \mathbf{p}' \in U$. We choose labelings of the three nodes for \mathbf{p} and \mathbf{p}' respectively, and let (X, j) and (X', j') be the associated marked $K3$ surfaces. Then the equality $\mathcal{P}(\mathbf{p}) = \mathcal{P}(\mathbf{p}')$ means that we have a Hodge isometry $\Phi: H^2(X, \mathbb{Z}) \rightarrow H^2(X', \mathbb{Z})$ with $\Phi \circ j = j' \circ i(\sigma)$ for some $\sigma \in \mathfrak{S}_3$. In particular, we have $\Phi(j(h)) = j'(h)$, $\Phi(j(f_j)) = j'(f_j)$, and $\Phi(j(e_i)) = j'(e_i)$ for $i \leq 8$. As before, we deduce that \mathbf{p} and \mathbf{p}' are PGL_3 -equivalent. This concludes the proof. \square

Corollary 7.2. *The variety $\mathcal{F}_{L_-}(\Gamma^+)$ is rational. Hence $\mathcal{M}_{(12,10,1)}$ is unirational.*

Remark 7.3. The space U/PGL_3 is birationally identified with the moduli of marked del Pezzo surfaces of degree 1. It would be interesting to study the rational action of the Weyl group on $\mathcal{F}_{L_-}(\Gamma^+)$ induced by the above immersion. Kondō [22] described the moduli of del Pezzo surfaces of degree 1 as a ball quotient.

7.2. The unirationality of $\widetilde{\mathcal{M}}_{(12,8,1)}$. Let $U \subset |\mathcal{O}_{\mathbb{P}^2}(3)| \times (\mathbb{P}^2)^8$ be the locus of cubics with points $(C, \mathbf{p}) = (C, p_1, \dots, p_8)$ such that (i) p_1, \dots, p_8 are distinct, (ii) C is smooth and passes $\{p_i\}_{i \neq 6}$, (iii) p_1, \dots, p_6 lie on a smooth conic Q , (iv) p_6, p_7, p_8 lie on a line L , and (v) $C + Q + L$ has only nodes as singularities. The sextic $C + Q + L$ is uniquely determined by (C, \mathbf{p}) . By setting $p_9 = L \cap C \setminus \{p_7, p_8\}$, $p_{10} = L \cap Q \setminus p_6$, and $p_{11} = Q \cap C \setminus \{p_i\}_{i=1}^5$, we have a complete marking of the nodes of $C + Q + L$. For the proof of unirationality it is convenient to reduce sextics with labelings to such cubics with points, and consider the space U of the latter.

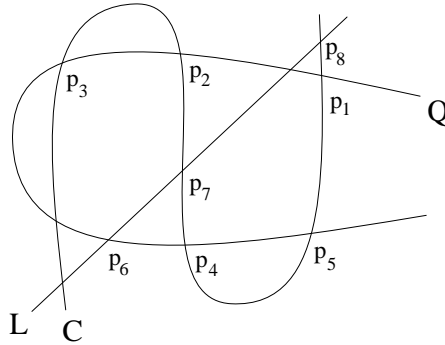


FIGURE 3. Sextic curve for $(r, a, \delta) = (12, 8, 1)$

Lemma 7.4. *The variety U is unirational.*

Proof. Let $V \subset (\mathbb{P}^2)^6$ be the locus of six points (p_1, \dots, p_6) lying on some conic and $W \subset (\mathbb{P}^2)^3$ be the locus of three collinear points (q_1, q_2, q_3) . The fiber product $V \times_{\mathbb{P}^2} W$ over $\mathbb{P}^2 = \{p_6 \in \mathbb{P}^2\} = \{q_1 \in \mathbb{P}^2\}$ is birational to the image of the projection $U \rightarrow (\mathbb{P}^2)^8$, $(C, \mathbf{p}) \mapsto \mathbf{p}$. As a general fiber of the projection $U \rightarrow V \times_{\mathbb{P}^2} W$ is an

open set of a plane in $|\mathcal{O}_{\mathbb{P}^2}(3)|$, it suffices to prove the unirationality of $V \times_{\mathbb{P}^2} W$, which is easily reduced to that of V . Let $p_1, \dots, p_4 \in \mathbb{P}^2$ be four general points and S be the blow-up of \mathbb{P}^2 at $\{p_i\}_{i=1}^4$. The conic pencil determined by $\{p_i\}_{i=1}^4$ defines a morphism $S \rightarrow \mathbb{P}^1$. We have a birational map $\mathrm{PGL}_3 \times (S \times_{\mathbb{P}^1} S) \dashrightarrow V$. Then the existence of sections of $S \rightarrow \mathbb{P}^1$ implies the rationality of $S \times_{\mathbb{P}^1} S$. \square

For a $(C, \mathbf{p}) \in U$, the 2-elementary $K3$ surface (X, ι) associated to the sextic $C + Q + L$ has main invariant $(12, 8, 1)$. As before, our labeling for $C + Q + L$ will induce an isometry $j: L_+ \rightarrow L_+(X, \iota)$ from an appropriate reference lattice L_+ . This defines a morphism $\tilde{p}: U \rightarrow \widetilde{\mathcal{M}}_{(12,8,1)}$, and we have the following.

Theorem 7.5. *The period map \tilde{p} descends to an open immersion $U/\mathrm{PGL}_3 \rightarrow \widetilde{\mathcal{M}}_{(12,8,1)}$ from a geometric quotient U/PGL_3 .*

Corollary 7.6. *For $a \leq 8$ the covers $\widetilde{\mathcal{M}}_{(12,a,\delta)}$ are unirational.*

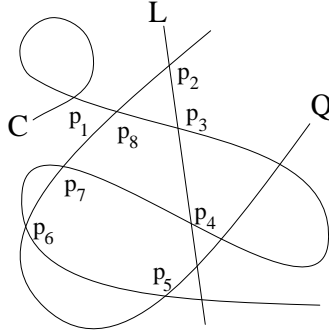
8. THE CASE $r = 13$

In this section we study the case $r = 13$. In Section 8.1 we construct a birational map from a configuration space of eight special points in \mathbb{P}^2 to a certain cover of $\mathcal{M}_{(13,9,1)}$ in a similar way as Section 7.1. In Section 8.2 we prove that the covers $\widetilde{\mathcal{M}}_{(13,a,\delta)}$ are unirational for $a \leq 7$.

8.1. $\mathcal{M}_{(13,9,1)}$ and eight special points in \mathbb{P}^2 . Let $M_{13} = \langle h, e_1, \dots, e_{12} \rangle$ be the lattice defined in (2.1). We define the vectors $f_1, f_2, f_3 \in M_{13}^\vee$ by $2f_3 = 3h - 2e_1 - \sum_{i=3}^{11} e_i$, $2(f_1 + f_2) = 3h - 2(e_2 + e_{12}) - \sum_{i=3}^{11} e_i$, and $2f_2 = 2h - (e_2 + e_{12}) - \sum_{i=5}^{10} e_i$. The overlattice $L_+ = \langle M_{13}, f_1, f_2, f_3 \rangle$ is 2-elementary of main invariant $(13, 9, 1)$. We let \mathfrak{S}_2 act on L_+ by the permutation on $\{e_9, e_{10}\}$. We fix a primitive embedding $L_+ \subset \Lambda_{K3}$ and set $L_- = (L_+)^\perp \cap \Lambda_{K3}$. The lattice L_- is isometric to $\langle 2 \rangle^2 \oplus \langle -2 \rangle^7$. Then let $\Gamma \subset \mathrm{O}(L_-)$ be the group $r_-^{-1}(\lambda \circ r_+(\mathfrak{S}_2))$, where $r_\pm: \mathrm{O}(L_\pm) \rightarrow \mathrm{O}(D_{L_\pm})$ and $\lambda: \mathrm{O}(D_{L_+}) \rightarrow \mathrm{O}(D_{L_-})$ are defined as in Section 7.1. The arithmetic quotient $\mathcal{F}_{L_-}(\Gamma^+)$ is a quotient of $\widetilde{\mathcal{M}}_{(13,9,1)}$ by \mathfrak{S}_2 , and dominates $\mathcal{M}_{(13,9,1)}$.

Let $V \subset (\mathbb{P}^2)^8$ be the codimension 1 locus of eight distinct points $\mathbf{p} = (p_1, \dots, p_8)$ such that (i) there exists an irreducible nodal cubic C passing $\{p_i\}_{i \neq 2}$ with $\mathrm{Sing}(C) = p_1$, (ii) p_2 lies on the line $L = \overline{p_3 p_4}$, (iii) there exists a smooth conic Q passing $\{p_2\} \cup \{p_i\}_{i=5}^8$, and (iv) the sextic $C + Q + L$ has only nodes as singularities. We shall denote $p_{11} = L \cap C \setminus \{p_3, p_4\}$ and $p_{12} = L \cap Q \setminus p_2$. In this way we obtain from \mathbf{p} the sextic $C + Q + L$ and the partial labeling $(p_1, \dots, p_8, p_{11}, p_{12})$ of its nodes. The remaining two nodes $\mathcal{S} = Q \cap C \setminus \{p_i\}_{i=5}^8$ are not naturally marked. We have a geometric quotient V/PGL_3 as a 7-dimensional variety, which is rational by Proposition 2.10.

For a $\mathbf{p} \in V$, let (X, ι) be the 2-elementary $K3$ surface associated to the sextic $C + Q + L$. A temporary choice of a labeling $\mathcal{S} = \{p_9, p_{10}\}$ induces a natural isometry $j: L_+ \rightarrow L_+(X, \iota)$, which defines a point in $\widetilde{\mathcal{M}}_{(13,9,1)}$ as the period of $((X, \iota), j)$. Considering the image in $\mathcal{F}_{L_-}(\Gamma^+)$ of the period of $((X, \iota), j)$, we obtain a well-defined morphism $\mathcal{P}: V \rightarrow \mathcal{F}_{L_-}(\Gamma^+)$ as in Section 7.1. Then we have the following.

FIGURE 4. Sextic curve for $(r, a, \delta) = (13, 9, 1)$

Theorem 8.1. *The period map \mathcal{P} descends to an open immersion $V/\mathrm{PGL}_3 \rightarrow \mathcal{F}_{L_-}(\Gamma^+)$. In particular, $\mathcal{F}_{L_-}(\Gamma^+)$ is rational and $\mathcal{M}_{(13,9,1)}$ is unirational.*

8.2. $\widetilde{\mathcal{M}}_{(13,7,1)}$ and pointed cubics. Let $U \subset |\mathcal{O}_{\mathbb{P}^2}(3)| \times (\mathbb{P}^2)^6$ be the space of pointed cubics $(C, \mathbf{p}) = (C, p_{1+}, p_{1-}, \dots, p_{3-})$ such that (i) C is smooth, (ii) p_{1+}, \dots, p_{3-} are distinct points on C , and (iii) if we denote $L_i = \overline{p_{i+}p_{i-}}$, the sextic $C + \sum_i L_i$ has only nodes as singularities. The variety U is rational, for the natural projection $U \rightarrow (\mathbb{P}^2)^6$ is birational to the projectivization of a vector bundle on an open set. For a pointed cubic $(C, \mathbf{p}) \in U$ we set $p_i = L_i \cap C \setminus \{p_{i+}, p_{i-}\}$ and $q_i = L_j \cap L_k$ where $\{i, j, k\} = \{1, 2, 3\}$. Thus we associate to (C, \mathbf{p}) the nodal sextic $C + \sum_i L_i$ with the labeling $(p_\mu, q_i)_{\mu,i}$ of its nodes. As before, from these we will obtain a lattice-marked 2-elementary $K3$ surface $((X, \iota), j)$ of type $(13, 7, 1)$. This defines a morphism $\tilde{p}: U \rightarrow \widetilde{\mathcal{M}}_{(13,7,1)}$, and we have the following.

Theorem 8.2. *The period map \tilde{p} descends to an open immersion $U/\mathrm{PGL}_3 \rightarrow \mathcal{M}_{(13,7,1)}$ from a geometric quotient U/PGL_3 .*

Corollary 8.3. *The covers $\widetilde{\mathcal{M}}_{(13,a,\delta)}$ for $a \leq 7$ are unirational.*

9. THE CASE $r \geq 14$

Let $U_d, V_d \subset (\mathbb{P}^2)^d$ be the loci defined in Section 2.3. By Proposition 2.10, when $d \geq 5$, we have geometric quotients U_d/PGL_3 and V_d/PGL_3 as rational varieties of dimension $2d - 8$ and $2d - 9$ respectively. In this section we prove the following.

Theorem 9.1. *One has birational period maps $U_d/\mathrm{PGL}_3 \dashrightarrow \widetilde{\mathcal{M}}_{(28-2d, 2d-6, \delta)}$ and $V_d/\mathrm{PGL}_3 \dashrightarrow \widetilde{\mathcal{M}}_{(29-2d, 2d-7, 1)}$ for $5 \leq d \leq 7$.*

By Proposition 3.10 and Figure 1 we have the following corollary, which completes the proof of Theorem 1.1.

Corollary 9.2. *The covers $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ are unirational for $r \geq 14$.*

Our constructions of the period maps are similar to those for eight points (Sections 7.1 and 8.1): we draw a sextic from a given point set, label its singularities in

a natural way, and then associate a lattice-marked 2-elementary $K3$ surface. Unlike the eight point cases, our labelings for $d \leq 7$ leave no ambiguity, and so we obtain points in $\widetilde{\mathcal{M}}_{(r,22-r,\delta)}$. Actually, these period maps may be derived from the ones for eight points by degeneration: as we specialize a configuration of points, the resulting sextic gets more degenerate, and the period goes to a Heegner divisor.

Theorem 9.1 for U_6 was first found by Matsumoto-Sasaki-Yoshida [26]. Considering degeneration, they essentially obtained the assertion also for V_6, U_5, V_5 with $\delta = 1$. The novelty of Theorem 9.1 is the constructions for $d = 7$. But even for $d \leq 6$, our period maps differ from the ones in [26]. Specifically, from a given point set we draw lines on the same plane, while in [26] the point set is regarded as a set of lines on the dual plane. Our argument as explained in Section 3.4 makes it easier to derive the monodromy groups, which were found by direct calculations in [26].

9.1. $\widetilde{\mathcal{M}}_{(14,8,1)}$ and seven general points in \mathbb{P}^2 . Let $U \subset (\mathbb{P}^2)^7$ be the open set of seven distinct points $\mathbf{p} = (p_1, \dots, p_7)$ such that (i) there exists an irreducible nodal cubic C passing p_1, \dots, p_7 with $\text{Sing}(C) = p_7$ and (ii) if we denote $L_i = \overline{p_i p_{i+3}}$ for $i \leq 3$, the sextic $C + \sum_i L_i$ has only nodes as singularities. We put $q_i = L_i \cap C \setminus \{p_i, p_{i+3}\}$ and $q_{ij} = L_i \cap L_j$. We thus obtain from \mathbf{p} the nodal sextic $C + \sum_i L_i$ and the complete labeling $(p_i, q_\mu)_{i,\mu}$ of its nodes. The components of $C + \sum_i L_i$ are also labelled obviously. Taking the right resolution of $C + \sum_i L_i$ and using these labelings, we obtain a lattice-marked 2-elementary $K3$ surface $((X, \iota), j)$ of type $(14, 8, 1)$ as before. This defines a morphism $\tilde{p}: U \rightarrow \widetilde{\mathcal{M}}_{(14,8,1)}$, and we will see the following.

Theorem 9.3. *The period map \tilde{p} descends to an open immersion $U/\text{PGL}_3 \rightarrow \widetilde{\mathcal{M}}_{(14,8,1)}$ from a geometric quotient U/PGL_3 .*

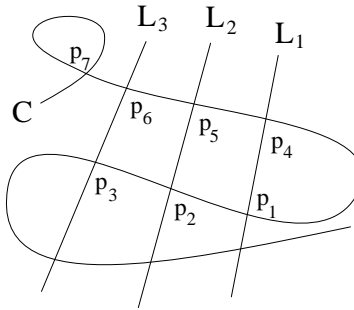


FIGURE 5. Sextic curve for $(r, a, \delta) = (14, 8, 1)$

In the next section we degenerate the points p_5, p_6, p_7 to collinear position. This forces the cubic C to degenerate to the union of a conic and a line.

9.2. $\widetilde{\mathcal{M}}_{(15,7,1)}$ **and seven special points in \mathbb{P}^2 .** Let $V \subset (\mathbb{P}^2)^7$ be the codimension 1 locus of seven distinct points $\mathbf{p} = (p_1, \dots, p_7)$ such that (i) p_5, p_6, p_7 lie on a line L_0 , (ii) p_1, \dots, p_4, p_7 lie on a smooth conic Q , and (iii) if we put $L_i = \overline{p_i p_{i+3}}$ for $1 \leq i \leq 3$, the sextic $Q + \sum_{i=0}^3 L_i$ has only nodes as singularities. We set $q_0 = L_0 \cap Q \setminus p_7$, $q_i = L_i \cap Q \setminus p_i$ for $i = 2, 3$, and $q_{ij} = L_i \cap L_j$ when $q_{ij} \neq p_k$ for some k . In this way we obtain from \mathbf{p} the sextic $Q + \sum_i L_i$, the labeling $(p_i, q_\mu)_{i,\mu}$ of its nodes, and also the obvious labeling of its components. As before, from these we obtain a lattice-marked 2-elementary $K3$ surface of type $(15, 7, 1)$. This defines a morphism $\tilde{p}: V \rightarrow \widetilde{\mathcal{M}}_{(15,7,1)}$, and we have the following.

Theorem 9.4. *The period map \tilde{p} descends to an open immersion $V/\mathrm{PGL}_3 \rightarrow \widetilde{\mathcal{M}}_{(15,7,1)}$ from a geometric quotient V/PGL_3 .*

In the next section we degenerate p_7 on $\overline{p_1 p_2}$. Then p_7 is determined as $\overline{p_1 p_2} \cap \overline{p_5 p_6}$, so that the parameters are reduced to six points. (We make renumbering).

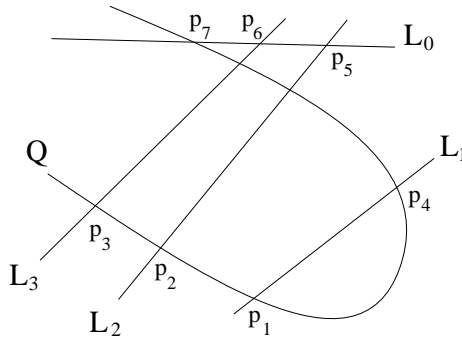
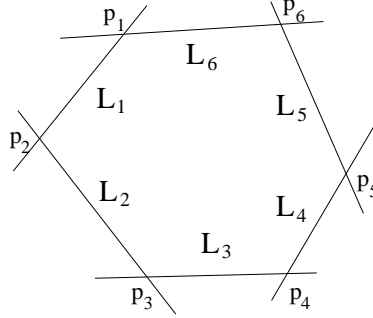


FIGURE 6. Sextic curve for $(r, a, \delta) = (15, 7, 1)$

9.3. $\widetilde{\mathcal{M}}_{(16,6,1)}$ **and six general points in \mathbb{P}^2 .** Let $U \subset (\mathbb{P}^2)^6$ be the open set of six distinct points $\mathbf{p} = (p_1, \dots, p_6)$ such that if we draw six lines by $L_1 = \overline{p_1 p_2}, \dots, L_5 = \overline{p_5 p_6}$, and $L_6 = \overline{p_6 p_1}$, then the sextic $\sum_i L_i$ has only nodes as singularities. Since the nodes of $\sum_i L_i$ are the intersections of the lines L_i , the labeling (L_1, \dots, L_6) of the lines induces that of the nodes, e.g., by setting $p_{ij} = L_i \cap L_j$. Hence from \mathbf{p} we obtain the sextic $\sum_i L_i$ with a labeling of its nodes and components. This defines a lattice-marked 2-elementary $K3$ surface of type $(16, 6, 1)$. Thus we obtain a morphism $\tilde{p}: U \rightarrow \widetilde{\mathcal{M}}_{(16,6,1)}$, and see the following.

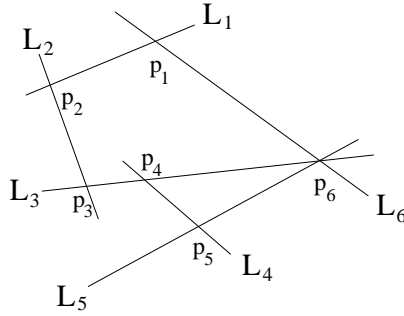
Theorem 9.5. *The period map \tilde{p} descends to an open immersion $\tilde{\mathcal{P}}: U/\mathrm{PGL}_3 \rightarrow \widetilde{\mathcal{M}}_{(16,6,1)}$ from a geometric quotient U/PGL_3 .*

Remark 9.6. If we identify $\mathbb{P}^2 \simeq |\mathcal{O}_{\mathbb{P}^2}(1)|$, the assignment $\mathbf{p} \mapsto (L_1, \dots, L_6)$ induces a Cremona transformation w of U/PGL_3 . The period map of [26] is written as $\tilde{\mathcal{P}} \circ w^{-1}$. One sees that w^2 is the cyclic permutation (654321) on U/PGL_3 .

FIGURE 7. Sextic curve for $(r, a, \delta) = (16, 6, 1)$

9.4. $\widetilde{\mathcal{M}}_{(17,5,1)}$ and six special points in \mathbb{P}^2 . Let $V \subset (\mathbb{P}^2)^6$ be the codimension 1 locus of six distinct points $\mathbf{p} = (p_1, \dots, p_6)$ such that (i) p_3, p_4, p_6 are collinear, and (ii) if we draw lines by $L_1 = \overline{p_1 p_2}, \dots, L_5 = \overline{p_5 p_6}$, and $L_6 = \overline{p_6 p_1}$, then any singularity of the sextic $\sum_i L_i$ other than p_6 is a node. The point p_6 is an ordinary triple point of $\sum_i L_i$. As in Section 9.3, we obtain a labeling of the nodes of $\sum_i L_i$ from the obvious one of the lines L_i . Denoting by q_i the infinitely near point of p_6 given by L_i for $i = 3, 5, 6$, we also obtain a labeling of the branches of $\sum_i L_i$ at p_6 . The 2-elementary $K3$ surface (X, ι) associated to the sextic $\sum_i L_i$ has main invariant $(17, 5, 1)$. Here we encounter a triple point, but we can proceed as before referring to Example 3.6: if $g: X \rightarrow \mathbb{P}^2$ is the natural projection branched over $\sum_i L_i$, the curve $g^{-1}(p_6)$ over p_6 consists of four labelled (-2) -curves, namely the (-2) -curves over q_i and a component of X^ι . Together with the above labeling for the nodes and the lines, this induces an isometry $j: L_+ \rightarrow L_+(X, \iota)$ from a reference lattice L_+ . Thus we obtain a morphism $\tilde{p}: V \rightarrow \widetilde{\mathcal{M}}_{(17,5,1)}$, and see the following.

Theorem 9.7. *The period map \tilde{p} descends to an open immersion $V/\mathrm{PGL}_3 \rightarrow \widetilde{\mathcal{M}}_{(17,5,1)}$ from a geometric quotient V/PGL_3 .*

FIGURE 8. Sextic curve for $(r, a, \delta) = (17, 5, 1)$

Degenerating p_2, p_4, p_5 to collinear position produces a period map for $\widetilde{\mathcal{M}}_{(18,4,0)}$ (Section 9.5), while degenerating p_4 to p_3 produces that for $\widetilde{\mathcal{M}}_{(18,4,1)}$ (Section 9.6).

9.5. $\widetilde{\mathcal{M}}_{(18,4,0)}$ and five general points in \mathbb{P}^2 . Let $U \subset (\mathbb{P}^2)^5$ be the open set of five distinct points $\mathbf{p} = (p_1, \dots, p_5)$ such that no three of p_1, \dots, p_5 other than $\{p_1, p_2, p_3\}$ are collinear. For a $\mathbf{p} \in U$ we draw six lines by $L_i = \overline{p_i p_4}$ for $1 \leq i \leq 3$ and $L_i = \overline{p_{i-3} p_5}$ for $4 \leq i \leq 6$. Then the sextic $\sum_{i=1}^6 L_i$ has ordinary triple points at p_4 and p_5 , nodes at $L_i \cap L_j$ for $i \leq 3$ and $j \geq 4$, and no other singularity. The obvious labeling of the lines L_i induces that of the nodes and the branches at the triple points of $\sum_i L_i$. The 2-elementary $K3$ surface (X, ι) associated to $\sum_i L_i$ has invariant $(r, a) = (18, 4)$. We have to identify its parity δ . Let (Y, B, π) be the right resolution of $\sum_i L_i$. We have the decomposition $B = \sum_{i=0}^7 B_i$ such that $\pi(B_i) = L_i$ for $1 \leq i \leq 6$ and $\pi(B_0) = p_5$, $\pi(B_7) = p_4$. One checks that the divisor $(\sum_{i=0}^3 B_i) - (\sum_{i=4}^7 B_i)$ is in $4NS_Y$. Hence (X, ι) has parity $\delta = 0$. Using our labeling for $\sum_i L_i$, we will obtain a morphism $\tilde{p}: U \rightarrow \widetilde{\mathcal{M}}_{(18,4,0)}$. Then we have the following.

Theorem 9.8. *The period map \tilde{p} descends to an open immersion $U/\mathrm{PGL}_3 \rightarrow \widetilde{\mathcal{M}}_{(18,4,0)}$ from a geometric quotient U/PGL_3 .*

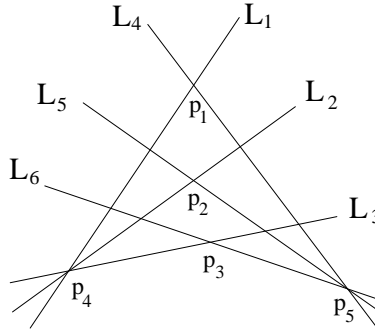


FIGURE 9. Sextic curve for $(r, a, \delta) = (18, 4, 0)$

9.6. $\widetilde{\mathcal{M}}_{(18,4,1)}$ and five general points in \mathbb{P}^2 . Let $U_5 \subset (\mathbb{P}^2)^5$ be the open set defined in Section 2.3. To a point $\mathbf{p} = (p_1, \dots, p_5)$ in U_5 we associate six lines by $L_1 = \overline{p_2 p_3}$, $L_i = \overline{p_1 p_{i+2}}$ for $i = 2, 3$, $L_i = \overline{p_i p_{i-2}}$ for $i = 4, 5$, and $L_6 = \overline{p_4 p_5}$. The sextic $\sum_i L_i$ has ordinary triple points at p_4 and p_5 . Any other singularity of $\sum_i L_i$ is a node. The 2-elementary $K3$ surface (X, ι) associated to $\sum_i L_i$ has invariant $(r, a) = (18, 4)$. In order to determine its parity δ , let $g: X \rightarrow \mathbb{P}^2$ be the natural projection branched over $\sum_i L_i$, and let E_{ij} be the (-2) -curves $g^{-1}(L_i \cap L_j)$ for $i, j \leq 3$. Then the \mathbb{Q} -divisor $D = \frac{1}{2}(E_{12} + E_{23} + E_{31})$ is in $L_+(X, \iota)^\vee$ by Proposition 3.2. Since $(D, D) = -\frac{3}{2}$, (X, ι) has parity $\delta = 1$. Using the obvious labeling of the lines L_i , we obtain a morphism $\tilde{p}: U_5 \rightarrow \widetilde{\mathcal{M}}_{(18,4,1)}$ as before. Then we see the following.

Theorem 9.9. *The period map \tilde{p} descends to an open immersion $U_5/\mathrm{PGL}_3 \rightarrow \widetilde{\mathcal{M}}_{(18,4,1)}$.*

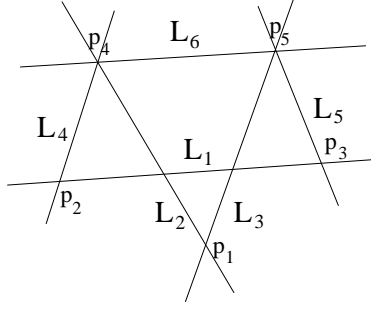


FIGURE 10. Sextic curve for $(r, a, \delta) = (18, 4, 1)$

9.7. $\widetilde{\mathcal{M}}_{(19,3,1)}$ and five special points in \mathbb{P}^2 . Let $V_5 \subset (\mathbb{P}^2)^5$ be the codimension 1 locus defined in Section 2.3. Given a point $\mathbf{p} = (p_1, \dots, p_5)$ in V_5 , for which p_1, p_2, p_3 are collinear, we define six lines in the same way as Section 9.6: $L_1 = \overline{p_2 p_3}$, $L_i = \overline{p_1 p_{i+2}}$ for $i = 2, 3$, $L_i = \overline{p_i p_{i-2}}$ for $i = 4, 5$, and $L_6 = \overline{p_4 p_5}$. Then the points p_1, p_4, p_5 are ordinary triple points of the sextic $\sum_i L_i$, and any other singularity of $\sum_i L_i$ is a node. As before, by taking the right resolution of the sextic $\sum_i L_i$ and using the labeling (L_1, \dots, L_6) of the lines, we obtain a lattice-marked 2-elementary K3 surface of type $(19, 3, 1)$. This defines a morphism $\tilde{p}: V_5 \rightarrow \widetilde{\mathcal{M}}_{(19,3,1)}$. Then we have the following.

Theorem 9.10. *The period map \tilde{p} descends to an open immersion $V_5/\mathrm{PGL}_3 \rightarrow \widetilde{\mathcal{M}}_{(19,3,1)}$.*

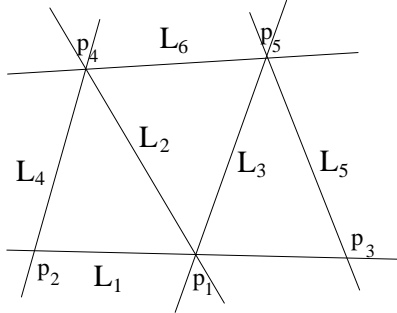


FIGURE 11. Sextic curve for $(r, a, \delta) = (19, 3, 1)$

10. MODULI OF BORCEA-VOISIN THREEFOLDS

The unirationality of $\mathcal{M}_{(r,a,\delta)}$ implies that of the moduli of Borcea-Voisin threefolds. Let (X, ι) be a 2-elementary $K3$ surface and E be an elliptic curve. The involution $(\iota, -1_E)$ of $X \times E$ extends to an involution j of the blow-up $\widetilde{X \times E}$ of $X \times E$ along the fixed curve of $(\iota, -1_E)$. The quotient $Z = \widetilde{X \times E} / \langle j \rangle$ is a smooth Calabi-Yau threefold ([35], [4]). The projection $\widetilde{X \times E} \rightarrow X$ (resp. $\widetilde{X \times E} \rightarrow E$) induces a fibration $\pi_1: Z \rightarrow Y = X / \langle \iota \rangle$ (resp. $\pi_2: Z \rightarrow E / \langle -1_E \rangle$) with constant E -fiber (resp. X -fiber), whose discriminant locus is the branch locus of the quotient morphism $X \rightarrow Y$ (resp. $E \rightarrow E / \langle -1_E \rangle$). Following [37], we call the triplet (Z, π_1, π_2) the *Borcea-Voisin threefold* associated to (X, ι) and E . Two Borcea-Voisin threefolds are isomorphic if and only if the corresponding 2-elementary $K3$ surfaces and elliptic curves are respectively isomorphic ([37]). The data (π_1, π_2) may be regarded as a kind of polarization of Z , as the following remark shows.

Lemma 10.1. *Let (Z, π_1, π_2) , (Z', π'_1, π'_2) be Borcea-Voisin threefolds, and let Λ (resp. Λ') be the primitive closure of $\pi_1^* \text{Pic} Y$ in $\text{Pic} Z$ (resp. $(\pi'_1)^* \text{Pic} Y'$ in $\text{Pic} Z'$). Then we have $(Z, \pi_1, \pi_2) \simeq (Z', \pi'_1, \pi'_2)$ if and only if we have $(Z, \Lambda) \simeq (Z', \Lambda')$.*

Proof. It suffices to prove the “if” part. Let $f: Z \rightarrow Z'$ be an isomorphism with $f^* \Lambda' = \Lambda$. There exist a very ample line bundle H on Y and a line bundle H' on Y' with $\pi_1^* H \simeq f^* (\pi'_1)^* H'$. Since $|H| \simeq |\pi_1^* H| \simeq |(\pi'_1)^* H'| \simeq |H'|$, we see that H' is base point free. Via the projective morphisms $Z \rightarrow |\pi_1^* H|^\vee$ and $Z' \rightarrow |(\pi'_1)^* H'|^\vee$, we obtain a morphism $g: Y' \rightarrow Y$ with $g \circ \pi'_1 = \pi_1 \circ f^{-1}$. One checks that g is bijective and hence is isomorphic. Considering the fibers and the discriminant loci of π_1 and π'_1 , we obtain $E \simeq E'$ and $(X, \iota) \simeq (X', \iota')$. \square

The main invariant of a Borcea-Voisin threefold is defined as that of the associated 2-elementary $K3$ surface. Obviously, two Borcea-Voisin threefolds are deformation equivalent if and only if they have the same main invariant. Let $X(1) = \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ be the moduli space of elliptic curves.

Theorem 10.2 ([37]). *The variety $\mathcal{M}_{(r,a,\delta)} \times X(1)$ is a coarse moduli space of Borcea-Voisin threefolds of main invariant (r, a, δ) .*

By Theorem 1.1 we have the following.

Theorem 10.3. *The moduli spaces of Borcea-Voisin threefolds are unirational.*

Acknowledgement. I would like to express my gratitude to my advisor, Professor Ken-Ichi Yoshikawa, for suggesting this problem and for his help and encouragement. I am further indebted to him for generously writing his work in the appendix. I am grateful to Dr. S. Okawa for helpful discussion on invariant theory. I also thank Professors S. Kondō, Y. Miyaoka, S. Mukai, and T. Terasoma for their comments. Finally, I would like to thank a referee for helpful suggestions for improving and simplifying the exposition. This work was supported by Grant-in-Aid for JSPS fellows [21-978].

APPENDIX A.

by Ken-Ichi Yoshikawa*

In this note, we give a proof of the following result using automorphic forms.

Theorem A.1. *The moduli space $\mathcal{M}_{(r,a,\delta)}$ has Kodaira dimension $-\infty$ if either $13 \leq r \leq 17$ or $r + a = 22$, $r \leq 17$.*

This is a consequence of the following criterion due to Gritsenko [12] (the idea first appeared in [14]).

Theorem A.2 (Gritsenko). *Let L be a lattice of signature $(2, n)$ with $n \geq 3$ and $\Gamma \subset \mathrm{O}(L)^+$ be a subgroup of finite index. Following [13], let $R \subset \Omega_L^+$ denote the ramification divisor of the projection $\pi: \Omega_L^+ \rightarrow \mathcal{F}_L(\Gamma)$. Suppose we have an integer $\nu \geq 0$ and an automorphic form F_k on Ω_L^+ for Γ of weight k such that $k \geq \nu n$ and that $\nu R - \mathrm{div}(F_k)$ is an effective divisor. If $k > \nu n$ or $\nu R - \mathrm{div}(F_k) \neq 0$, then*

$$\kappa(\mathcal{F}_L(\Gamma)) = -\infty.$$

Proof. When $\nu = 1$, the result is exactly [12, Th. 1.5]. When $\nu > 1$, the same proof works after replacing F_{nm}/F_k^m by F_{nm}^ν/F_k^m in the proof of [12, Th. 1.5]. For the convenience of the reader, we give some detail. Assume $\omega \in H^0(\mathcal{F}_L(\Gamma), mK_{\mathcal{F}_L(\Gamma)})$, $m > 0$. Regard Ω_L^+ as a tube domain of \mathbb{C}^n . Then $\pi^*\omega = F_{nm}(z)(dz_1 \wedge \dots \wedge dz_n)^{\otimes m}$, where $F_{nm}(z)$ is a non-zero automorphic form on Ω_L^+ for Γ of weight mn . Since ω is holomorphic on $\mathcal{F}_L(\Gamma)$, F_{nm} must vanish on R at least of order m (cf. [13]). Hence $\mathrm{div}(F_{nm}) - mR \geq 0$. Then F_{nm}^ν/F_k^m is an automorphic form for Γ of weight $-m(k - \nu n) \leq 0$ with effective divisor

$$\mathrm{div}(F_{nm}^\nu/F_k^m) \geq m(\nu R - \mathrm{div}(F_k)) \geq 0.$$

Since $n \geq 3$, F_{nm}^ν/F_k^m must be a constant. Hence $k = \nu n$ and $\nu R = \mathrm{div}(F_k)$, which contradicts the assumption. \square

As an application of his criterion, Gritsenko gives several examples of orthogonal modular varieties with Kodaira dimension $-\infty$. See [12] for those examples. We thank Professor V.A. Gritsenko, whose lecture in the conference “Moduli and Discrete Groups” at RIMS, Kyoto (2009) inspired this note and who kindly showed his paper [12] when we wrote this note.

A.1. The case $13 \leq r \leq 17$.

Theorem A.3. *If $13 \leq r \leq 17$, then $\kappa(\mathcal{M}_{(r,a,\delta)}) = -\infty$.*

Proof. Let L_- be the anti-invariant lattice of a 2-elementary K3 surface of type (r, a, δ) with $r \geq 11$. We denote $g = 11 - \frac{1}{2}(r + a)$. By [38, Th. 8.1], there exists an automorphic form Ψ_{L_-} for $\mathrm{O}(L_-)^+$ of weight $k = (r - 6)(2^g + 1)$ with divisor $\mathrm{div}(\Psi_{L_-}) = D'_{L_-} + (2^g + 1)D''_{L_-}$, where D'_{L_-} and D''_{L_-} are reduced divisors

$$D'_{L_-} := \sum_{\lambda \in L_-, \lambda^2 = -2, \lambda/2 \notin L_-^\vee} \lambda^\perp, \quad D''_{L_-} := \sum_{\lambda \in L_-, \lambda^2 = -2, \lambda/2 \in L_-^\vee} \lambda^\perp.$$

*Research partially supported by the Grants-in-Aid for Scientific Research (B) 19340016, JSPS

By definition, $D := D'_{L_-} + D''_{L_-}$ is the discriminant divisor of $\Omega_{L_-}^+$. Let $R \subset \Omega_{L_-}^+$ be the ramification divisor of the projection $\Omega_{L_-}^+ \rightarrow \mathcal{F}(\mathcal{O}(L_-)^+)$. We set $\nu = 2^g + 1$ in Theorem A.2. Since $n = 20 - r$ and $r \geq 13$, we get $k - \nu n = 2\nu(r - 13) \geq 0$. Since $R \geq D$ by [13, Proof of Th. 1.1.], we get $\nu R - \text{div}(\Psi_{L_-}) \geq (\nu - 1)D'_{L_-} \geq 0$. When $r > 13$ or $D'_{L_-} \neq 0$, the result follows from Theorem A.2. When $r = 13$ and $D'_{L_-} = 0$, then $L_- = U(2) \oplus M_7$. Let $r \in L_-$ be a vector with $r^2 = -4$. Since the reflection with respect to r is an element of $\mathcal{O}(L_-)^+$, we get $r^\perp \subset R$ and $r^\perp \not\subset D$, which implies $\nu R - \text{div}(\Psi_{L_-}) \neq 0$. The result follows again from Theorem A.2. \square

A.2. The case $r + a = 22$ and $r \leq 17$. We construct an automorphic form for $\mathcal{O}(L_-)^+$ satisfying the conditions in Theorem A.2 as a Borchers product [5]. For this, we first construct a modular form of type ρ_{L_-} with those properties required in [5, Th. 13.3]. In what follows, we write $r_- = r(L_-)$, $a_- = a(L_-)$, $\sigma_- = 4 - r_-$. Let $\text{Mp}_2(\mathbb{Z})$ be the metaplectic double cover of $\text{SL}_2(\mathbb{Z})$, which is generated by $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}$ and $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1$. See [5, Sect. 2] for more about $\text{Mp}_2(\mathbb{Z})$.

A.2.1. Elliptic modular forms. We set $q = e^{2\pi i \tau}$ for $\tau \in \mathbb{H}$ and

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \theta_{\langle 2 \rangle}(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}, \quad \theta_{\langle 2 \rangle + 1/2}(\tau) = \sum_{n \in \mathbb{Z}} q^{(n + \frac{1}{2})^2}.$$

Set $\text{M}\Gamma_0(4) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \in \text{Mp}_2(\mathbb{Z}); c \equiv 0 \pmod{4} \}$. By [6, Lemma 5.2], there exists a character $\chi_\theta: \text{M}\Gamma_0(4) \rightarrow \{\pm 1, \pm i\}$ such that $\theta_{\langle 2 \rangle}(\tau)$ is a modular form for $\text{M}\Gamma_0(4)$ of weight $1/2$ with character χ_θ .

Set $\eta_{1-82^{84-8}}(\tau) := \eta(\tau)^{-8} \eta(2\tau)^8 \eta(4\tau)^{-8}$ and define $\psi_m(\tau)$, $m \in \mathbb{Z}$, by

$$\psi_m(\tau) := \eta_{1-82^{84-8}}(\tau)^2 \theta_{\langle 2 \rangle}(\tau)^{8+m} - 2(m+16) \eta_{1-82^{84-8}}(\tau) \theta_{\langle 2 \rangle}(\tau)^m.$$

Since $\eta_{1-82^{84-8}}(\tau)$ is a modular form for $\text{M}\Gamma_0(4)$ of weight -4 with trivial character, $\psi_m(\tau)$ is a modular form for $\text{M}\Gamma_0(4)$ of weight $\frac{m-8}{2}$ with character χ_θ^m . Since $\eta_{1-82^{84-8}}(\tau) = q^{-1} + 8 + 36q + O(q^2)$ and $\theta_{\langle 2 \rangle}(\tau) = 1 + 2q + O(q^4)$, we get

$$\psi_m(\tau) = q^{-2} + 2(-m^2 - 9m + 124) + O(q).$$

Write $\psi_m(\tau) = \sum_{l \in \mathbb{Z}} d_m(l) q^l$ and define $h_m^{(i)}(\tau)$, $i \in \mathbb{Z}/4\mathbb{Z}$ as the series

$$h_m^{(i)}(\tau) := \sum_{l \equiv i \pmod{4}} d_m(l) q^{l/4}.$$

Then we have $\sum_{i \in \mathbb{Z}/4\mathbb{Z}} h_m^{(i)}(\tau) = \psi_m(\tau/4)$.

A.2.2. Vector-valued elliptic modular forms. Let $\mathbb{C}[D_{L_-}]$ be the group ring of the discriminant group D_{L_-} with the standard basis $\{\mathbf{e}_\gamma\}_{\gamma \in D_{L_-}}$. The Weil representation $\rho_{L_-}: \text{Mp}_2(\mathbb{Z}) \rightarrow \text{GL}(\mathbb{C}[D_{L_-}])$ is defined as follows (cf. [5, Sect. 2]):

$$\rho_{L_-}(T) \mathbf{e}_\gamma := e^{\pi i \gamma^2} \mathbf{e}_\gamma, \quad \rho_{L_-}(S) \mathbf{e}_\gamma := \frac{i^{-\sigma_-/2}}{|D_{L_-}|^{1/2}} \sum_{\delta \in D_{L_-}} e^{-2\pi i \langle \gamma, \delta \rangle} \mathbf{e}_\delta.$$

We use the notion of modular forms of type ρ_{L_-} , for which we refer to [5, Sect. 2].

Our construction is based on the following observation due to Borchers.

Proposition A.4. *If $\phi(\tau)$ is a modular form for $\mathrm{M}\Gamma_0(4)$ with character $\chi_\theta^{\sigma_-}$, then*

$$\mathcal{B}_{L_-}[\phi](\tau) := \sum_{g \in \mathrm{M}\Gamma_0(4) \backslash \mathrm{M}\mathrm{p}_2(\mathbb{Z})} \phi|_g(\tau) \rho_{L_-}(g^{-1}) \mathbf{e}_0$$

is a modular form for $\mathrm{M}\mathrm{p}_2(\mathbb{Z})$ of type ρ_{L_-} of the same weight as that of $\phi(\tau)$, where $\phi|_g(\tau) := \phi\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-2l}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}\mathrm{p}_2(\mathbb{Z})$.

Proof. See e.g. [38, Prop. 7.1]. \square

Set $V := S^{-1}T^2S = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \sqrt{-2\tau+1}$. The coset $\mathrm{M}\Gamma_0(4) \backslash \mathrm{M}\mathrm{p}_2(\mathbb{Z})$ is represented by $\{1, S, ST, ST^2, ST^3, V\}$. We define $\mathbf{v}_k := \sum_{\delta \in D_{L_-}, \delta^2 \equiv k/2 \pmod{2}} \mathbf{e}_\delta \in \mathbb{C}[D_{L_-}]$ for $k \in \mathbb{Z}/4\mathbb{Z}$. Let $\mathbf{1}_{L_-} \in D_{L_-}$ be the unique element such that $\langle \mathbf{1}_{L_-}, \gamma \rangle = \gamma^2 \pmod{\mathbb{Z}}$ for all $\gamma \in D_{L_-}$. By [38, Proof of Lemma 7.5], we get the following relations

$$\begin{aligned} \rho_{L_-}((ST^l)^{-1}) \mathbf{e}_0 &= i^{\frac{\sigma_-}{2}} 2^{-\frac{a_-}{2}} \sum_{k=0}^3 i^{-lk} \mathbf{v}_k, & \rho_{L_-}(V^{-1}) \mathbf{e}_0 &= \mathbf{e}_{\mathbf{1}_{L_-}}, \\ \eta_{1^{-8}2^{8}4^{-8}}|_{ST^l}(\tau) &= 2^4 \eta_{1^{-8}2^{8}4^{-8}}\left(\frac{\tau+l}{4}\right), & \eta_{1^{-8}2^{8}4^{-8}}|_V(\tau) &= -16\eta(2\tau)^{-16}\eta(4\tau)^8, \\ \theta_{\langle 2 \rangle}|_{ST^l}(\tau) &= (2i)^{-\frac{1}{2}} \theta_{\langle 2 \rangle}\left(\frac{\tau+l}{4}\right), & \theta_{\langle 2 \rangle}|_V(\tau) &= \theta_{\langle 2 \rangle+1/2}(\tau). \end{aligned}$$

Then we get

$$\psi_m|_{ST^l}(\tau) = 2^{\frac{8-m}{2}} i^{-\frac{m}{2}} \psi_m\left(\frac{\tau+l}{4}\right).$$

Since $\eta(2\tau)^{-16}\eta(4\tau)^8 = 1 + O(q)$ and $\theta_{\langle 2 \rangle+1/2}(\tau) = 2q^{1/4} + O(q^{5/4})$, we get

$$\psi_m|_V(\tau) = O(q^{m/4}).$$

In what follows, we assume $r_- < 12$ and $m = 8 + \sigma_-$. Then

$$\begin{aligned} \sum_{l=0}^3 \psi_m|_{ST^l}(\tau) \rho_{L_-}((ST^l)^{-1}) \mathbf{e}_0 &= 2^{-\frac{\sigma_-+a_-}{2}} \sum_{j=0}^3 \sum_{l=0}^3 \sum_{s \in \mathbb{Z}/4\mathbb{Z}} h_m^{(s)}(\tau+l) i^{-lj} \mathbf{v}_j \\ &= 2^{\frac{r_-+a_-}{2}} \sum_{j=0}^3 h_m^{(j)}(\tau) \mathbf{v}_j. \end{aligned}$$

By Proposition A.4, $\mathcal{B}_{L_-}[\psi_{8+\sigma_-}]$ is a modular form of type ρ_{L_-} of weight $\sigma_-/2$. By the definition of $\mathcal{B}_{L_-}[\psi_{8+\sigma_-}]$ and the expansion of $h_m^{(l)}(\tau)$, we get the expansion

$$\begin{aligned} \mathcal{B}_{L_-}[\psi_{8+\sigma_-}](\tau) &= \psi_{8+\sigma_-}(\tau) \mathbf{e}_0 + 2^{\frac{r_-+a_-}{2}} \sum_{l=0}^3 h_{8+\sigma_-}^{(l)}(\tau) \mathbf{v}_l + \psi_{8+\sigma_-}|_V(\tau) \mathbf{e}_{\mathbf{1}_{L_-}} \\ &= \left\{ q^{-2} + 2(-m^2 - 9m + 124) + O(q) \right\} \mathbf{e}_0 \\ &\quad + 2^{\frac{r_-+a_-}{2}} \left\{ 2(-m^2 - 9m + 124) + O(q) \right\} \mathbf{v}_0 + O(q^{1/4}) \mathbf{v}_1 \\ &\quad + 2^{\frac{r_-+a_-}{2}} \{ q^{-1/2} + O(q^{1/2}) \} \mathbf{v}_2 + O(q^{3/4}) \mathbf{v}_3 + O(q^{m/4}) \mathbf{e}_{\mathbf{1}_{L_-}}. \end{aligned}$$

From the first equality, we see that $O(L_-)$ preserves $\mathcal{B}_{L_-}[\psi_{8+\sigma_-}]$ (cf. [38, Th. 7.7 (2)]). By [5, Th. 13.3], the Borchers lift $\Xi_{L_-} := \Psi_{L_-}(\cdot, \mathcal{B}_{L_-}[\psi_{8+\sigma_-}])$ is a holomorphic automorphic form on $\Omega_{L_-}^+$ for $O(L_-)^+$ of weight $(2^{\frac{r_- - a_-}{2}} + 1)(-m^2 - 9m + 124)$ with zero divisor

$$\operatorname{div}(\Xi_{L_-}) = \sum_{\lambda \in L_-, \lambda^2 = -4} \lambda^\perp + 2^{\frac{r_- - a_-}{2}} \sum_{\lambda \in L_-^\vee, \lambda^2 = -1} \lambda^\perp.$$

Theorem A.5. *If $r + a = 22$ and $11 \leq r \leq 17$, then $\kappa(\mathcal{M}_{(r,a,\delta)}) = -\infty$.*

Proof. By the conditions $r + a = 22$ and $11 \leq r \leq 17$, we get $r_- = a_-$ and $5 \leq r_- \leq 11$. We have an explicit expression $L_- = \langle 2 \rangle^2 \oplus \langle -2 \rangle^{r_- - 2}$, from which we get $L_-^\vee = \frac{1}{2}L_-$. We set $\mathcal{H} := \sum_{\lambda \in L_-, \lambda^2 = -4} \lambda^\perp$. Then $\operatorname{div}(\Xi_{L_-}) = 2\mathcal{H}$. If $\lambda \in L_-$ and $\lambda^2 = -4$, then the reflection with respect to λ is an element of $O(L_-)^+$. Hence we get the inclusion of divisors $R \supset \mathcal{H}$, which implies $R - \mathcal{H} \geq 0$.

We set $v = 1$, $k = -m^2 - 9m + 124$ and $F_k = \Xi_{L_-}^{1/2}$ in Theorem A.2. Since $n = r_- - 2$, we get $k - n = -m^2 - 8m + 114 > 0$ when $r_- \geq 5$, i.e., $m \leq 7$. Since $\operatorname{div}(F_k) = \mathcal{H}$, we get $R - \operatorname{div}(F_k) \geq 0$. Now the result follows from Theorem A.2. \square

REFERENCES

- [1] Alexeev, V.; Nikulin, V. V. *Del Pezzo and K3 surfaces*. MSJ Memoirs **15**, 2006.
- [2] Baily, W. L., Jr.; Borel, A. *Compactification of arithmetic quotients of bounded symmetric domains*. Ann. Math. (2) **84** (1966) 442–528.
- [3] Barth, W. P.; Hulek, K.; Peters, C. A. M.; Van de Ven, A. *Compact complex surfaces*. Springer-Verlag, 2004.
- [4] Borcea, C. *K3 surfaces with involution and mirror pairs of Calabi-Yau manifolds*. Mirror symmetry, II, 717–743, 1997.
- [5] Borchers, R.E. *Automorphic forms with singularities on Grassmanians*. Invent. Math. **132** (1998), 491–562.
- [6] Borchers, R.E. *Reflection groups of Lorentzian lattices*. Duke Math. J. **104** (2000), 319–366.
- [7] Borel, A. *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*. J. Diff. Geom. **6** (1972), 543–560.
- [8] Coolidge, J. L. *A treatise on algebraic plane curves*. Dover Publications, 1959.
- [9] Dolgachev, I. V. *Mirror symmetry for lattice polarized K3 surfaces*. J. Math. Sci. **81** (1996), no. 3, 2599–2630.
- [10] Dolgachev, I. V.; van Geemen, B.; Kondō, S. *A complex ball uniformization of the moduli space of cubic surfaces via periods of K3 surfaces*. J. Reine Angew. Math. **588** (2005), 99–148.
- [11] Dolgachev, I. V.; Ortland, D. *Point sets in projective spaces and theta functions*. Astérisque **165** (1988).
- [12] Gritsenko, V. *Reflective modular forms in algebraic geometry*. arXiv:1005.3753.
- [13] Gritsenko, V.A., Hulek, K., Sankaran, G.K. *The Kodaira dimension of the moduli of K3 surfaces*. Invent. Math. **169** (2007), 519–567.
- [14] Gritsenko, V.A., Hulek, K., Sankaran, G.K. *Hirzebruch-Mumford proportionality and locally symmetric varieties of orthogonal type*. Doc. Math. **13** (2008), 1–19.
- [15] Harris, J. *On the Severi problem*. Invent. Math. **84** (1986), no. 3, 445–461.
- [16] Horikawa, E. *Surjectivity of the period map of K3 surfaces of degree 2*. Math. Ann. **228** (1977), no. 2, 113–146.
- [17] Hunt, B.; Weintraub, S. H. *Janus-like algebraic varieties*. J. Diff. Geom. **39** (1994), no.3, 509–557.

- [18] Huybrechts, D.; Stellari, P. *Proof of Căldăraru's conjecture*. Appendix: "Moduli spaces of twisted sheaves on a projective variety" by K. Yoshioka. Moduli spaces and arithmetic geometry, 31–42, Adv. Stud. Pure Math. **45**, 2006.
- [19] Kharlamov, V. M. *Topological types of nonsingular surfaces of degree 4 in RP^3* . Funct. Anal. Appl. **10** (1976), 295–305.
- [20] Koike, K.; Shiga, H.; Takayama, N.; Tsutsui, T. *Study on the family of K3 surfaces induced from the lattice $(D_4)^3 \oplus \langle -2 \rangle \oplus \langle 2 \rangle$* . Internat. J. Math. **12** (2001), no. 9, 1049–1085.
- [21] Kondō, S. *The rationality of the moduli space of Enriques surfaces*. Compositio Math. **91** (1994), no. 2, 159–173.
- [22] Kondō, S. *The moduli space of curves of genus 4 and Deligne-Mostow's complex reflection groups*. Algebraic geometry 2000, Azumino, 383–400, Adv. Stud. Pure Math., **36**, 2002.
- [23] Kondō, S. *The moduli space of 5 points on \mathbb{P}^1 and K3 surfaces*. Arithmetic and geometry around hypergeometric functions, 189–206, Progr. Math. **260**, 2007.
- [24] Kondō, S. *Moduli of plane quartics, Gopel invariants and Borcherds products*. Int. Math. Res. Notices (2011), 2825–2860.
- [25] Ma, S. *Rationality of the moduli spaces of 2-elementary K3 surfaces*. arXiv:1110.5110.
- [26] Matsumoto, K.; Sasaki, T.; Yoshida, M. *The monodromy of the period map of a 4-parameter family of K3 surfaces and the hypergeometric function of type (3, 6)*. Internat. J. Math. **3** (1992), no. 1.
- [27] Morrison, D. R.; Saitō, M.-H. *Cremona transformations and degrees of period maps for K3 surfaces with ordinary double points*. Algebraic geometry, Sendai, 1985, 477–513, Adv. Stud. Pure Math. **10**, 1987.
- [28] Mumford, D.; Fogarty, J.; Kirwan, F. *Geometric invariant theory*. Third edition. Springer-Verlag, 1994.
- [29] Nakayama, N. *Classification of log del Pezzo surfaces of index two*. J. Math. Sci. Univ. Tokyo **14** (2007), no. 3, 293–498.
- [30] Nikulin, V.V. *Integral symmetric bilinear forms and some of their applications*. Math. USSR Izv. **14** (1980), 103–167.
- [31] Nikulin, V.V. *Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections*. J. Soviet Math. **22** (1983), 1401–1476.
- [32] Shah, J. *A complete moduli space for K3 surfaces of degree 2*. Ann. Math. (2) **112** (1980), no. 3, 485–510.
- [33] Shah, J. *Degenerations of K3 surfaces of degree 4*. Trans. Amer. Math. Soc. **263** (1981), no. 2, 271–308.
- [34] Shepherd-Barron, N.I. *Invariant theory for S_5 and the rationality of M_6* . Compositio Math. **70** (1989), no. 1, 13–25.
- [35] Voisin, C. *Miroirs et involutions sur les surfaces K3*. Journées de Géométrie Algébrique d'Orsay (Orsay, 1992). Astérisque **218** (1993), 273–323.
- [36] Yoshikawa, K.-I. *K3 surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space*. Invent. Math. **156** (2004), no.1, 53–117.
- [37] Yoshikawa, K.-I. *Calabi-Yau threefolds of Borcea-Voisin, analytic torsion, and Borcherds products*. From Probability to Geometry (II), volume in honor of J.-M. Bismut, Astérisque **328** (2009) 355–393.
- [38] Yoshikawa, K.-I. *K3 surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space II*. arXiv:1007.2830, to appear in J. Reine. Angew. Math.

(S. Ma) GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO, TOKYO 153-8914, JAPAN

E-mail address: sma@ms.u-tokyo.ac.jp

(K.-I. Yoshikawa) DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN
KOREA INSTITUTE FOR ADVANCED STUDY, HOEGIRO 87, DONGDAEMUN-GU, SEOUL 130-722, KOREA

E-mail address: yosikawa@math.kyoto-u.ac.jp